Probability and Random Variables

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Lecture - 7
Function of a Random Variable

In the last class, we have discussed topics upto conditional distribution and conditional density function. In this class, we will consider what is called total probability and Bayes theorem after that may be an example will be taken problem. And then I will switch over to another topic, a new topic which is called function of a random variable. We will start that topic today.

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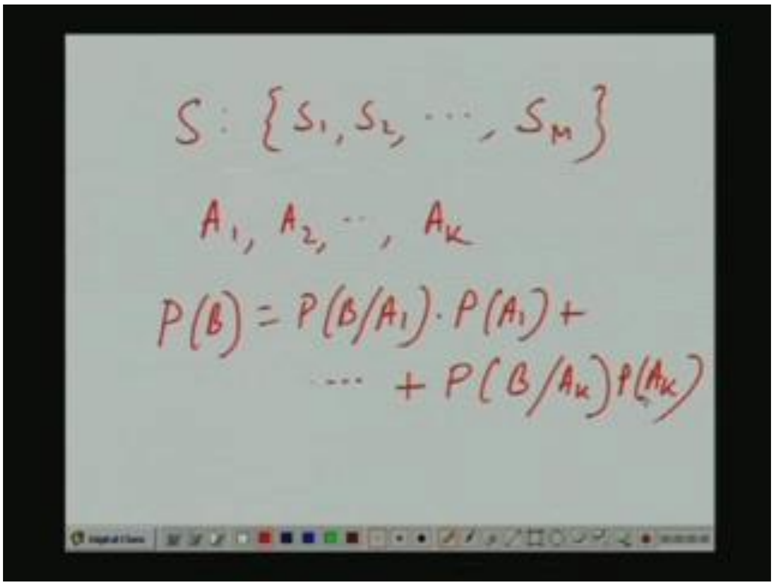

Suppose, S stands for the set of all possible outcomes and outcomes are like this; S1 S2 dot dot dots say SM, there are M outcomes. Now, if we form, we already know that I mean, any subset of these constitutes an event; suppose we consider some events A1 A2 dot dot dot say AK, which forms a part partition of S, that is A1 A2 and AK they are mutually disjoint and their union is equal to S. In that case, as you have seen earlier that for any arbitrary event B, we can always write P B as P B by A1 that is; probability of event B condition to the event A1 times P A1 plus dot dot dot dot that is, likewise P B by

AK P AK. Well there are for just probability discrete probabilities, but we can use the same concept to derive the, I mean corresponding case for probability distributions.

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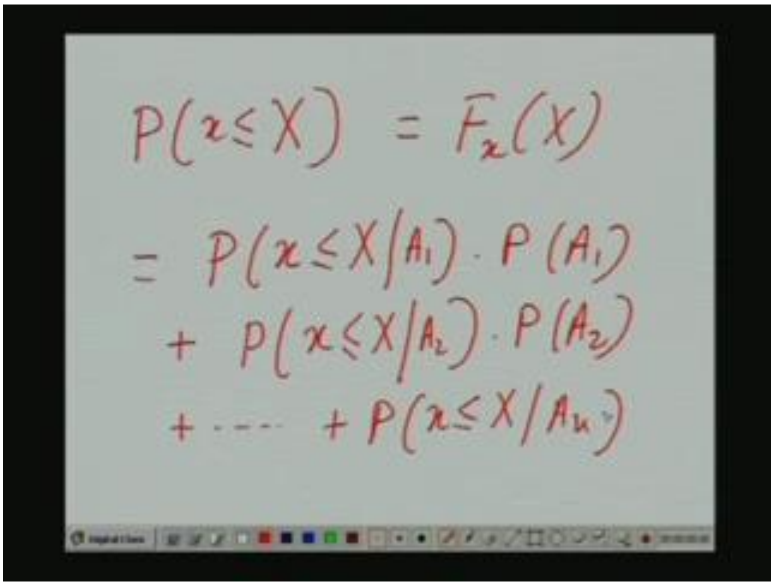

Suppose, it is like this; x is the random variable. S as I told you is a set of all possible outcomes and suppose there are K partitions A1 A2 up to AK. Then following the same procedure, we can write that, what is this? This stands for probability of a random variable x, taking values less than equal to some giving number X. Actually, this x less than equal to X denotes an event that is; it is set of all those outcomes, for which a random variable takes values less than equal to some pre-specified value X effect. This is what we call, as we recall Fx X there is probability distribution of the random variable x. Now, we can write this as, there is nothing new in this, I mean; we have just discussed the case for discrete probabilities and you know, we are just extending the extending it the concept here.

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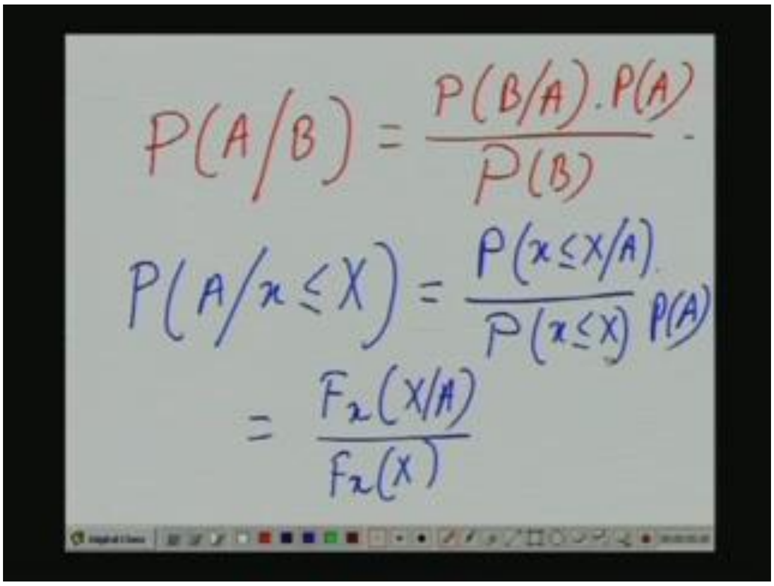

Similarly, we can write, we know that you can always write this thing, the P A by B is same as P B by A into P A divided by P B, this was discussed earlier. Again we can use the same concept, in the case of distributions, we can write P. But, this is nothing but, conditional distribution of x subject to the event A. This is nothing but the probability distribution x. So, this times P A. These are very useful results, which you come across in various applications of statistics. We just consider example here. And I am following the book by Papoulis, as I told in the beginning and it is a problem from Papoulis.

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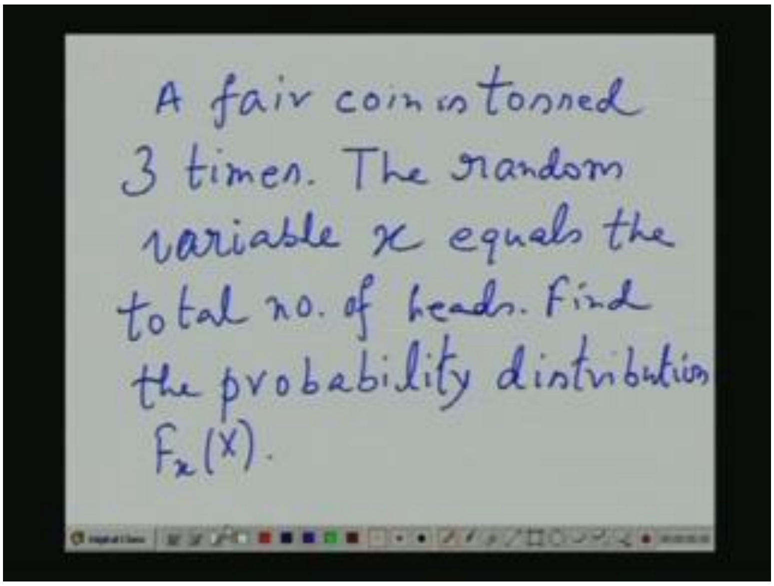

This problem says that, a fair coin tossed 3 times, fair coin means as you do its unbiased. The random variable equals the total number of heads. Obviously; x can be either 0, means in no occasion you got a head x can be either equal to 1 means 1 of the 3 tosses, give rise to head other to tails and likewise. So, the question asked is; find the probability, so we will consider all the possibilities. See there are 3 tosses 3 trials. At this time there are 2 possibilities; either head or tail. So, you can have 2 the power 3 that is, 8 possibilities.

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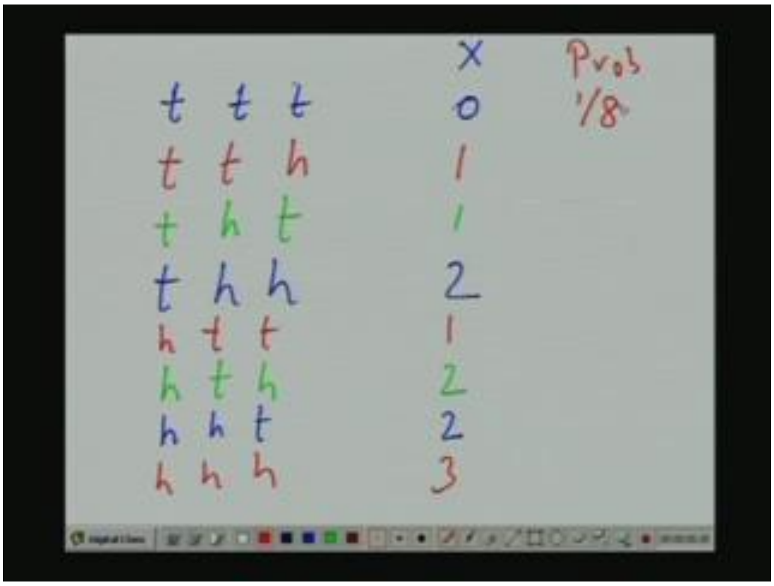

So, let us enlist them, one is t t t, in that case random variable takes the value a 0. Then, t t h, it takes the value 1. Then, t h t, again it takes of value 1, it is value 2 because, 2 h's have come, then again h t t 1 2 2, finally; 3. So, the corresponding probabilities, it is not distribution, just the probability. This is a fair coin right, this is unbiased. So, probability of t, occurring on the fast first of this is half, here also half, here also half. So, this probability is 1 by 8.

So, how about these? Here h occurs once and t h t, they occur twice. In this case what is the probability? I think it point be good idea to leave this problem at this stage. If I just given a hint and may be towards the end I will take it up, because I find that you know, I am not giving any problem for you, you to take up. So, I have gone up this. So, I give some time and you proceed with this, then we will take this up and we will work it out. You can easily start from here. Actually, the other reason why I did not pursue this is, this you know for time being short, I wanted to complete this topic and I wanted give a good coverage of this topic.

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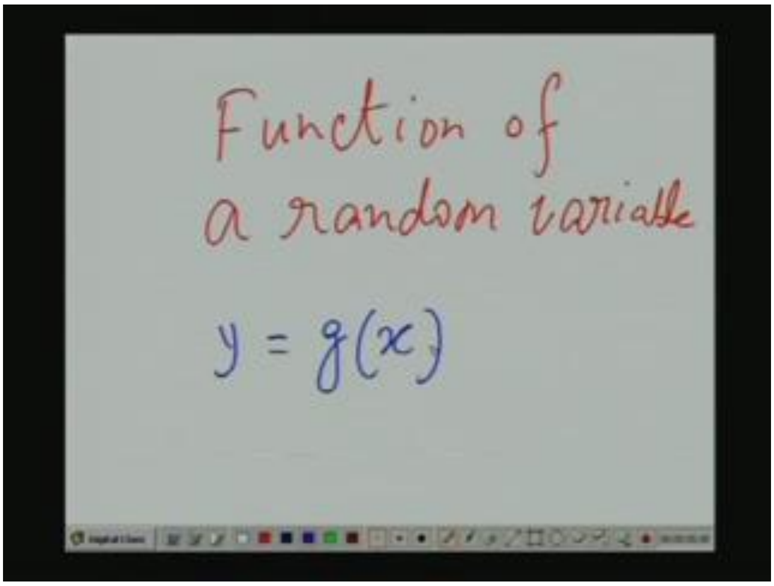

Function of suppose x is random variable, it means that, some experiment is going on and there are similar experimental outcomes. So, the total set is S. The various outcomes are S1 S2 S3; likewise with each outcome, I mean with the experimental outcomes, a random variable x is associated, so that for each outcome it takes a value. So, this is just repetition of what is the actual meaning or mathematical interpretation of this term; random variable.

Now, suppose I define a function; y equal to g x where g is a function. What does it mean? It means that, given any particular experimental outcome, for that outcome this x take some value and g is function which works on that value and gives you another out value which for y. That means, to tell you more precisely; suppose there is a experimental outcome.

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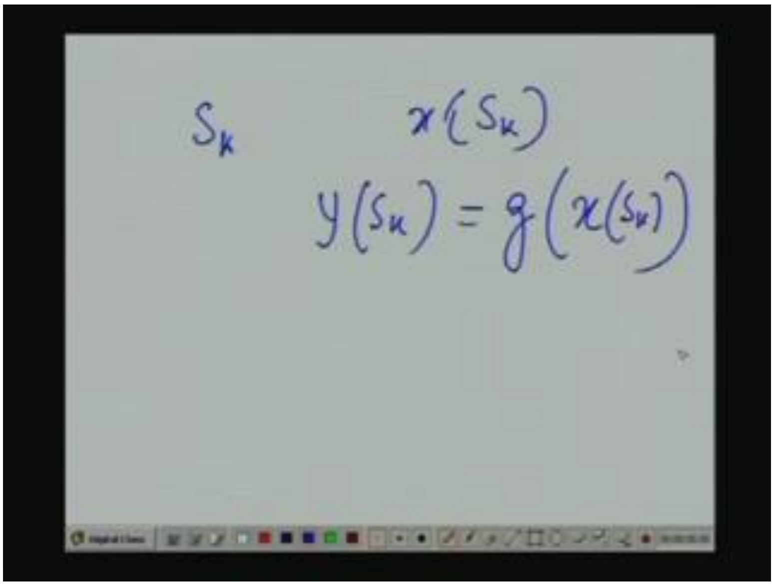

Suppose, there is an experiment outcome say SK; for this the random variable, x takes of value all right for this. How is y obtained? We say y. In fact, we should write y SK because, y also is random variable now. So, given the outcome experimental outcome SK for is value for y is nothing but that is for the outcome SK, find out the value x of SK and g works on that that is value for y. Question is; if we are given the probability distribution function for x how to find out the corresponding probability distribution function for y? That is a question.

But before that one thing is clear here, that if x a is random variable; obviously, y also is a random variable. Just there is some mathematical conditions that is required, that I just read out from this book, that is this they call that this function g should be a be Baire function.

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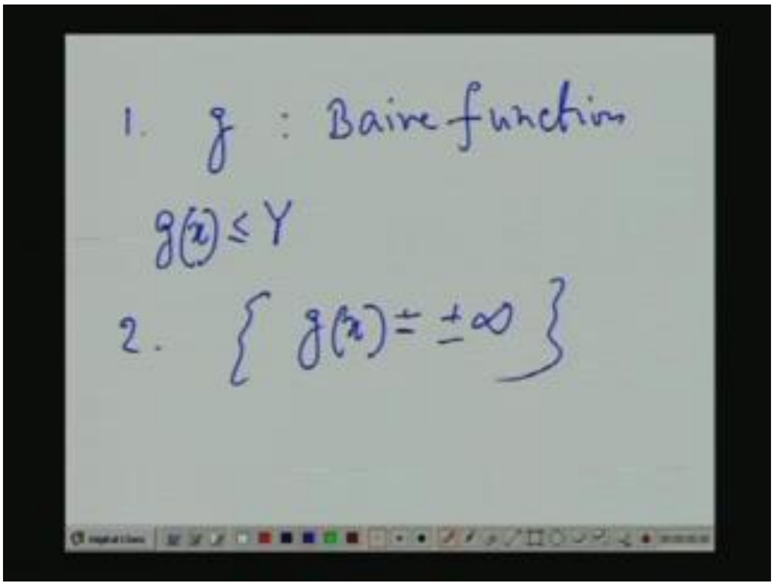

Meaning that given the value Y for the random variable Y, if you have to find out x a range of x so that, g x is less than equal to Y. And the range should consist of countable, the union and intersection of a countable number of intervals on the x axis. That is, if you take the x axis, it is not that there will be a continuous range, there can be various ranges and sub ranges, but total should be a countable number of union and intersection of intervals.

And the other thing is, this events should have 0 probability; obviously, this was 1 of the condition for the condition for the random variable, that is, we should not have any outcome for which value of Y assumes for which assumes the value infinity. So, that is why 0 probability. So, this event, for which g x is plus minus infinity, should have 0 probability. Then, this y will be actually in a strict sense a random variable. Now, how to find out the probability distribution of the function Y? For that, let consider the what is meant by Fy Y

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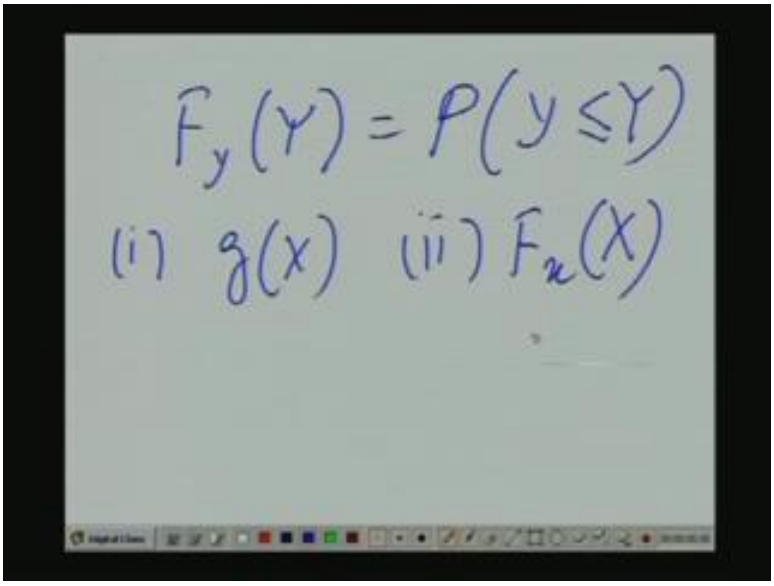

It is nothing but the probability of this event, but Y is some given number. This we know, this is very standard, this is the probability distribution of the random variable y. Now, we have to find it out given the function g and the probability distribution of x that is; we are given these 2 things: this function g and this is given to us. How to go about it? It is not very difficult, actually it is whether we physically we can easily, I mean visualize it. We are given some number Y. We know this function g X.

So, given this number Y, we will try to find out for what possible values of the random variable x, the corresponding function g of X, takes values less than equal to y. We have to find out the probability of that. That will give us this function. We start with Y and we know that g of X is always y. So, we have to find out those ranges of x, for which g of X takes values less than equal to y. And then, you have to find out the probability of those total probability of those values of x. That is how we should go about. So, I will illustrate this through examples. Then, we will derive some particular relation formula and all that.

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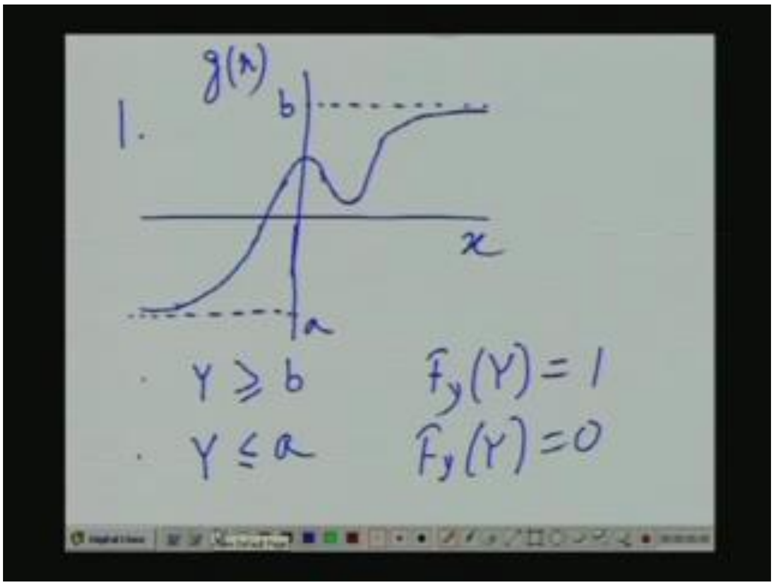

First consider a function like this. This is g x which is equal to Y. Now, you have to find out the probability distribution Fy Y. First suppose, Y is by the way look at this function, this is asymptotically touching b, it is not going above b, neither it is going below a. So, let us start like this; if Y x is given to be greater than or equal to b, then it is a very simple situation. The entire range of x, gives rise to values g x which lie below Y because, this is not touching b, it is only asymptotically touching b. So, if Y is either equal to b or above b.

Then for any x in this axis, corresponding function will remain below Y and the probability of x in this total range, from minus infinity to infinity that is, the total event actually because, this cannot take any other value that will be 1. So, in this case it is very simple, Fy Y will be equal to 1. On the other head, you consider this, suppose Y is less than equal to a. Now, here this function is asymptotically touching a, as we go to minus infinity as x goes to minus infinity. But is never really touching a or never really going below a.

So, if you place at Y, if you take Y below a, then this function remains above this for the entire range of x, from minus to infinity, it never goes below. In that case Fy Y will be 0 because, after all what is this? This means the probability of y taking values less than equal to Y. Y is either a less than a, at Y there is g x is never going below that. So, the corresponding probability will be 0. These are very simple cases. Then in fact, this is a new page. So, I have to redraw the figure again.

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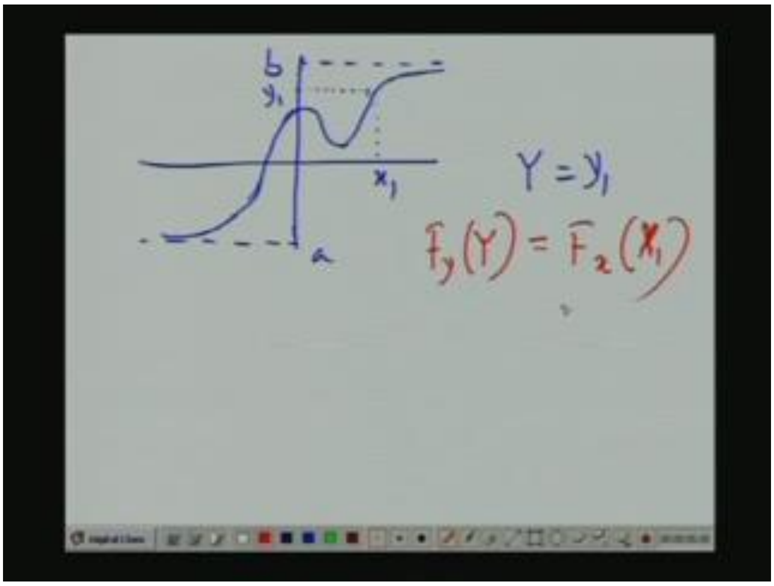

So, I do it quickly. Now, suppose I take a value here y1 and corresponding value here is x1. Suppose, Y is y 1, in that case what is Fy Y? Think for a moment and tell me what is Fy Y? Fy Y means probability of this y taking values less than equal to Y that is y 1. That is y should, y should be either y1 for below this. But you see here, I have y1 corresponding value of x is x1, at if you look towards the left of this x1. You see for each x, you get some value for y, but each value of such y is below y1.

So, to the end, so the entire left portion of this point, x1 will give rise to values of y less than equal to Y, this very clear from the figure. So; that means, this probability is nothing but, this. But, things may not always be so simple. Suppose, I take another value now, therefore I draw another figure because, these becoming clumsy.

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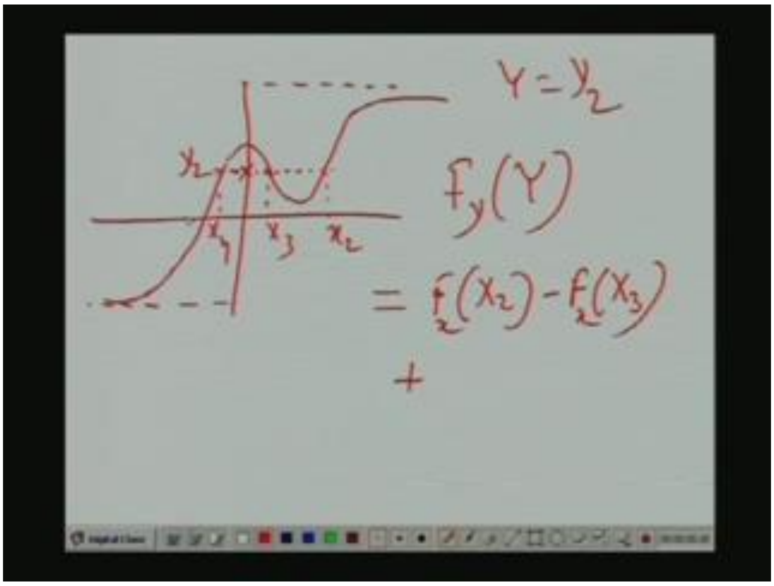

I take a value say here call it y 2. Now, here you see, this corresponds to 3 values of x, this corresponds to 3 values of x. 1 is here, another here, another here. You can call it x2 x3 x4. In this case what is Fy Y, where Y equal to y2, y2 is shown here. You see, we have to go below y2. Now, as you go below y2, you see if you are in this range, but between x2 to x3. I have this portion of the curve which is clearly below y2. So, this is an acceptable range, but x4 to x3 is not, because within this range, I have this portion of the curve which lies, above y2. But to the left of x4 if you go, again the curve falls below y2.

So; that means, I have to find out the 2 ranges for x, which are applicable here are this that is, 1 between x2 and x3 or x3 and x2, another to the left of x4, to the left of x4. So, this; that means, I have this probability is nothing but the probability of x lie within this range or lying to left of x4. Now, the probability of x, like in this range between x2 and x3 is, this we all know F X2 minus F X3, that is what is this after all? This is a probability distribution of x that is, the probability of x taking values less than equal x2.

Similarly, probability of x taking value less than equal to x3, their difference gives the probability of x taking values within this range. And to that, I have to add this F X4 and likewise. You can consider various ranges, I have arbitrarily put y above first above b, then below a, then somewhere here, then somewhere here. I will, through this example I am trying to show that, given such function you have to inspect the thing clearly. You consider all possible ranges and then only you can find out the actual description, of the corresponding probability distribution function f of Y. I take further examples of this kind a very simple example.

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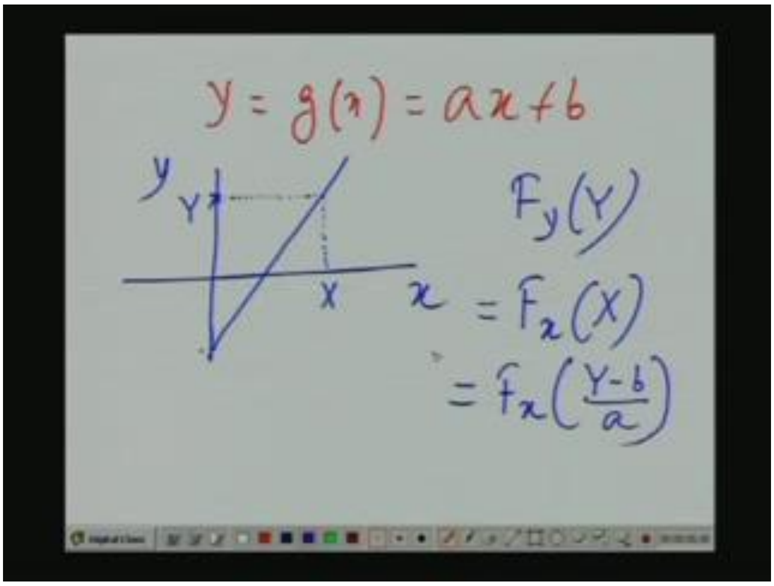

Suppose, y is given as g x as ax plus b, a very simple relation, that is a relationship of this straight line. So, suppose I plot it, it is like this. This much is b, we could there is nothing particular, about point using b 2 with negative, it could be here also. I have to find out the probability distribution of this variable y, this function y, this random variable y, that is, I have to find out as before this.

Suppose, I place my Y here, this gives rise to only 1 point in x because; these are linear relationship that makes life very easy here. Suppose, the corresponding point is X, then what is the probability distribution? It means the probability of Y taking values, Y taking values less than or equal to Y; that means, I have to go down from this point. But since, it is a straight line you can see here, that if I go below Y; that means, I go to the left of X.

So; that means, this probability is nothing but then this is expressed in terms of x, where as I want an expression in terms of Y, but is this is quite clearly Fx. Now X is what, if you put X see here, but you starting point was Y, so Y; that means, what is X? Then in terms of given Y what is this point? We can easily see from this equation. The distribution is in terms of x. As I told you in the beginning, there are 2 things are given to us: 1 is the function g x, another the probability distribution of the random variable x. Only thing is I want the probability distribution of Y as a function of y or Y, so first I am finding out what is this.

This in this case is F X, where X is related to Y like this. Then, I am replacing X, by the corresponding value of Y, because Y is given. I am evaluating this function for a particular Y, so this is this. But you know the interesting thing is, I mean if you stop here, then we only half do the job. As I told in the previous example, we have to really inspect the cases thoroughly and consider all possibilities. So, far, so good I mean, there is no ambiguity here, but we will see again things can change. If I just, you know I mean change some parameters, as well as change just orientation, little bit let me let me explain what I mean. Suppose in this equation y equal to ax plus b, this variable a is negative. In that case, you have an equation like this; this is y this is x, a is negative.

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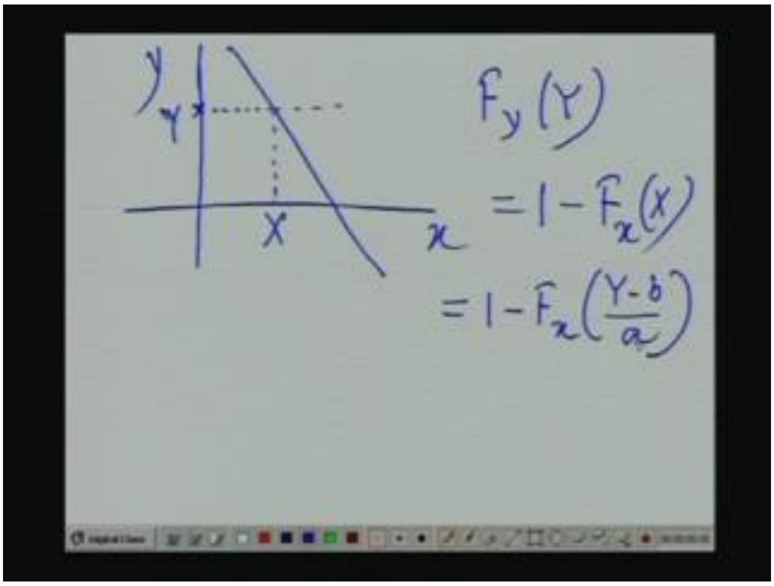

So, if I now start with a particularly Y, find out the possible X, then you see after all, our task is to find out this; that means, the probability of y taking values, less than equal to Y. But, if we go down, go below Y in this case, then unlike the previous case, we should not go to the left of X, because as you try to go to the left of X, this part of the straight line comes up, which lies above Y, because slope is negative now.

This means, I mean this means of probability distribution of Y which means the probability of y taking values less than or equal to Y. So, I have to go below, to satisfy this definition I have to go below from Y. But if the previous case, that meant that, we should go to the left of this point X. If this example that does not apply, because if you now go to the left of X, for each X the straight line I mean you can find the value of corresponding y, but that lies above this Y, the straight line that pervades above this Y. So, I cannot I do not have that range now.

On the other hand, if I go to the right of X, the whole straight line remains below Y. So; that means, I have to consider the range of x, starting at X and going to the right. What is that? That is nothing but 1 minus, after all what is Fx X? Fx X means the probability of x taking values less than or equal to X. That is the probability of the event x, falling in this entire range, to the left of X including X or to its left, 1 minus that will be the probability, of x following to the right of X. And as before, this is given in terms of x, but you know I mean I want a function in terms of y. I want a probability distribution function of y, I mean as a function of y itself. So, I replace X by this. We take another example.

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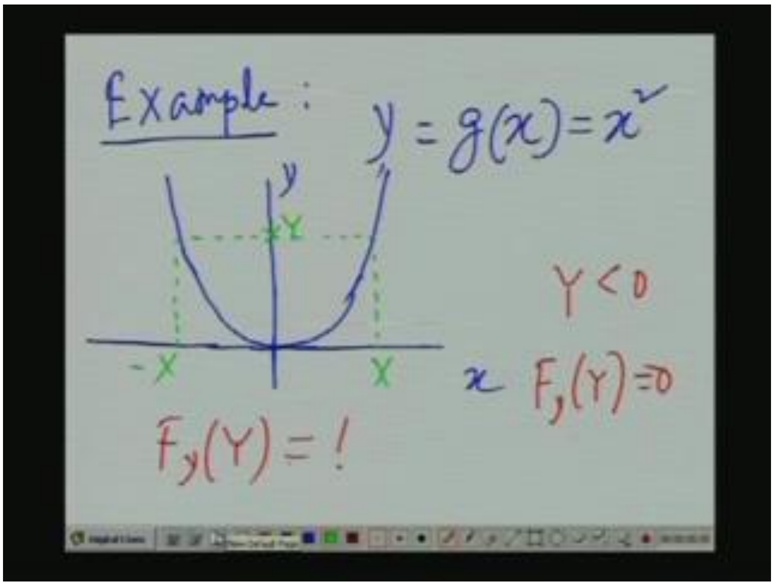

y equal to g x equal to x square, like this. We have to find out, now suppose what if I wrote these now suppose, Y has value less than equal to 0 that is, either at origin or less than that. In fact, I should say Y less than 0, not less than equal, but less than 0. In that case clearly, I mean there is no range of x for which Y takes value less than 0. So, in that case, on the other hand, if you have some value of Y, so I have got 2 values of X. In fact, 1 is X another minus X from symmetry upwards. And what is X? That is square root Y and this is minus square root Y. So, what will you have probability of Y taking values less than equal to capital Y? That is same as the probability of x taking values, within the range minus X to X.

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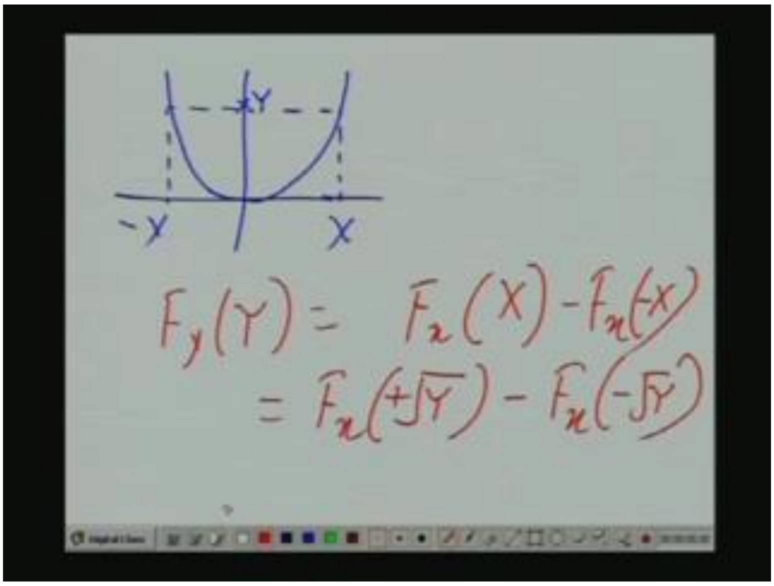

That means will be quickly redraw again this was here Y, this is X, this is minus X. In this case what is the probability of X lying within this range? There is very simple Fx X minus Fx minus X. And what is X? This square root to Y, I put a plus sign here and then minus square root Y. So, through this example, we will see that we really have to inspect the function. It has got various ranges, then certain parameter might change, instead of you know taking a positive value; it can take a negative value the whole shape of the function changes. You have to consider all the possibilities to come out with the probability distribution of the function that is Y. We still consider more cases.

Now, suppose I consider a case where we are given y equal to g x, but there is range of x for which, suppose this takes up a constant value. So, it has kind of you know, I mean some kind of something like a table or you know DC, it is region where it is it is you know it has a fixed value over a range. There is something, that something interesting occurs, that is for the other ranges of x, there is no problem, we can proceed as you have been proceed as you have been carrying on.

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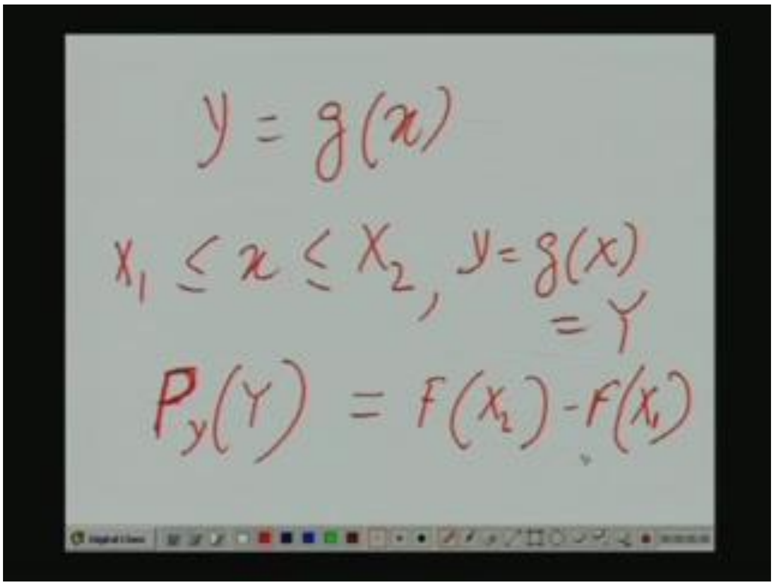

But we now suppose are given this, that for x less than equal to say X1 and greater than equal to X2, y equal to g x equal some Y, that is within this range, Y is fixed. For this particular Y what is the probability? In fact, it will be better if you just consider this, instead of distribution; I am not considering distribution, what is the probability of of y taking the value Y. That will be better. What is a probability of y taking the value Y. That will be same as the probability of x falling in this range and what is that probability?

Now, you see it has got an implication on the probability of distribution Fy Y. What is that implication? Suppose Y is fixed such that, you know Y is below, I mean this value. Suppose, I take a value of y Y prime which is the less than Y and for that, I have got some probability distribution function. As soon as I reach this Y, in the distribution function there will be a discontinuity, because a total of this much will get added to the previous probability distribution function. Let me explain through some examples. Actually in short whatever to the say is that, at this point Y these points.

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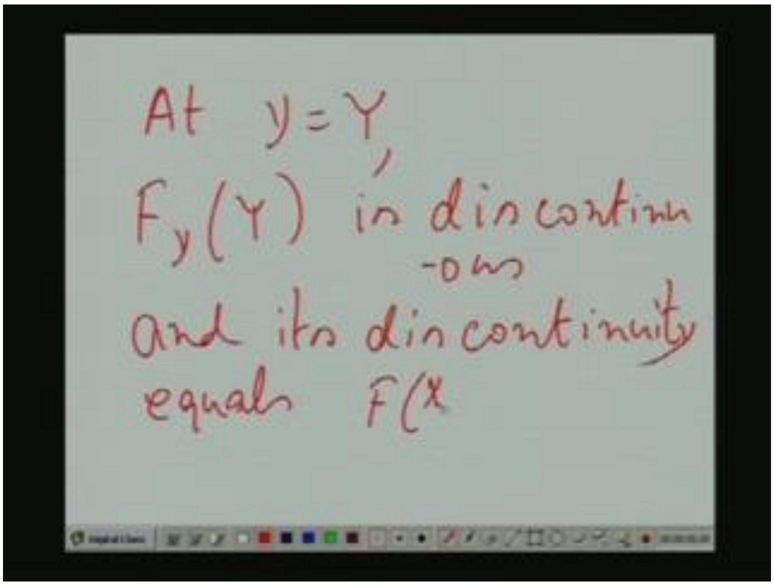

Physically you can those also. Suppose, I started with a very small value of this Y and as it as I take higher and higher value, this probability distribution increases, because it is after all a non decreasing function as you all know. Then the moment I reach here, then since the function g x was static that time, it was constant, this amount of this is an extra amount that gets added to that probability distribution function. And then as y moves above this Y, I will get continue to get this probability distribution function in a non decreasing manner. May be I take an example. Consider this case, a very simple case.

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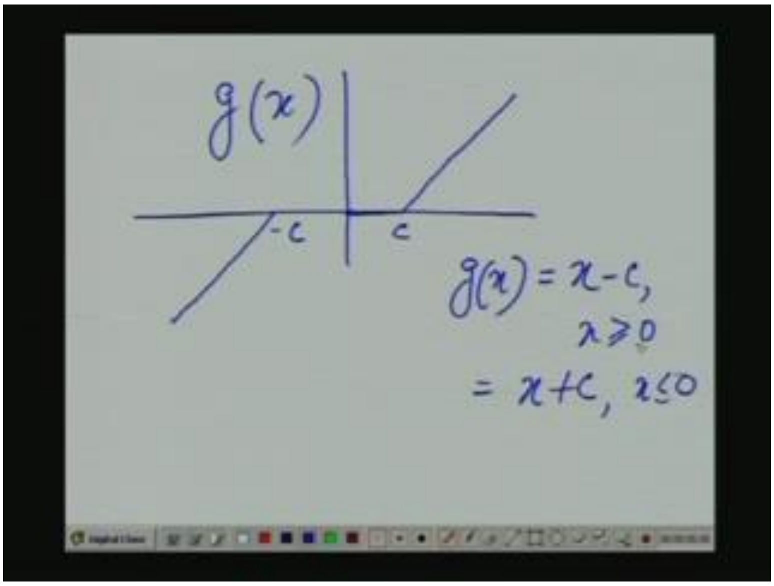

It is given g x is x minus c for x greater than equal to 0 and equal to x plus c for x less than equal to 0. In fact, to even be correct, you should write 0 plus and 0 minus to the to indicate the right side of the 0 and left side of 0. Interestingly you see this function has 1 zone from minus c to c, where it has got a constant value of the value of 0. This is given and other thing that is given is, I go to the next page something like this.

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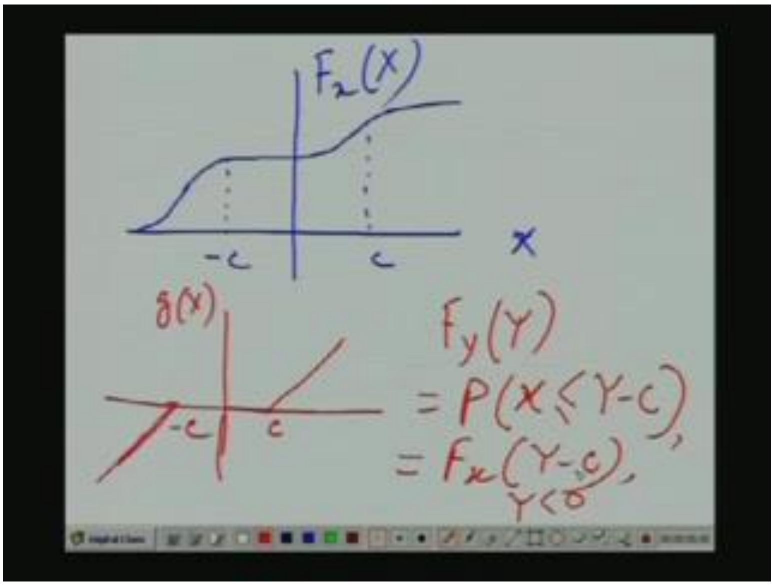

This is given; you have to find out the probability distribution function of y. Now, just for the sake of convenience, let us draw the figure here again for g x and we are finding out this. Suppose, Y is to the left of this side, left of minus c, Y is above below 0. Suppose Y below 0; that means, we are on this side, to the left of I mean, the range of x following to the left of minus c. For all them, I have got this the function g x given by this line, which is below this axis.

So; that means, if Y is less than 0, the corresponding probability distribution will be what? The net probability of x falling in this range, that is, after all the straight line equation is x plus c add Y remaining in this range means, X should be less than equal Y minus c. In fact, this is nothing but Fx Y minus c this is for Y less than 0. For Y less than 0, this is the thing. But what happens while Y is greater than 0? That time, again I redraw these figures, so that you know I do not have to go back to the previous page.

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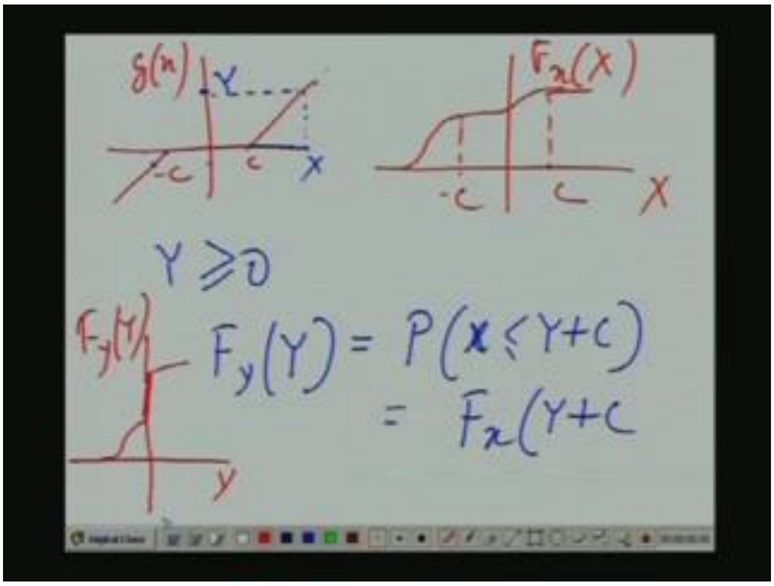

If this case, if Y is greater than equal to 0, then I am to the right of c. So, the probability of x following to the right c what is that? In fact, it is not like that, if I giving a particular value say Y, Y is here, earlier I took Y to the below 0 say Y was here. So, you know corresponding x was here and as I go below Y, the entire range to the left was considered; like that I derived the previous probability distribution function.

Now, suppose Y is above 0, somewhere here, then I am here. I have to find out the probability of x falling in this range, either at this point or to the left of this, if you call it X. So, the point is if you consider this total range, the probability of x falling in this range, you know that will be related to probability distribution of X. Now, if this X were here or here or here or here, then it simply becomes of probability distribution of X.

But, as we cross this, as we cross this I mean, X increased, but Y did not increase, some extra component came in. In fact, if you really if work it out, it will be like this because, here that straight line equation is x is minus c. So, X point expressed in terms Y is you know Y plus c. So, X would be less than equal to Y plus c, give in Y corresponding point is Y plus c. So, x would be less than equal to that, and that is this.

So, if you plot it. So, if this function is give in Fx, using that you can plot and instead of going to another page, I can carry out the plot here only. If you plot it, will be like this. This portion will be taken up, for y going up to 0 and then from 0 upwards from 0 onwards, I mean for values greater than equal to 0, portion to the right will be taken up, 0 should be nothing but nothing like this in, there is discontinuity and they this portion will come out. This is your Fy Y.

So, in this class, we have we just concluded our previous topic, that is conditional probability distribution and density functions. We give the expressions for total probability distribution and of course, if you derive that with respect to x, if you differentiate that with respect to x, you get the corresponding total probability density also. You know, we took up an example, but I did not carry out, I left it to you. In the next class I will take it up.

So, I hope you also will be able to come with the solution by that time, then subsequently, I took up a new topic all together, which is called function of a random variable. We defined what is meant by function of we explained what is mean by function of a random variable. Then our purpose was to; obtain the probability distribution function of that random variable y, which is written as a function of give a random variable x. Of course the distribution was carried out, was obtained from the knowledge of the function g x and from the knowledge of the probability distribution of the random variable x itself.

We considered various cases and various possibilities and in the next class will continue from here and we will try to the obtain a general formula, you know of obtaining the probability distribution function, of the probability distribution, of the function Y itself, expressed as a function of Y. So, that is all for today.

Thank you.

Probability and Random Variables

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Lecture - 8

Function of a Random Variable (Contd.)

You know in the previous class, we considered a problem. Then I was working it out, but then I left it half way through, I thought you would give it a try. So, let us solve that problem first and then we will consider the subsequent topic. The problem that I gave was this; that there is a fair coin, means the probability of having head or tail, either case is half. And then coin is tossed thrice and the random variable X equals the total number of heads. So, I have to find out the probability.