Probability & Random Variables
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Lecture - 6
Conditional Distribution and Density Functions

In the last class, we discussed you know probability distribution and probability density function and discussed the properties of probability distribution function. And then we considered some examples of probability density function like you know I mean normal that is: Gaussian density. Then binomial distribution that was the distribution and we considered the corresponding density also probability density function also. Then we considered binomial.

We considered uniform probability density and all that. We in today's class, we start from that. We consider 1 more probability distribution function, which is very useful especially in queuing theory and network related problems that is called Poisson distribution.

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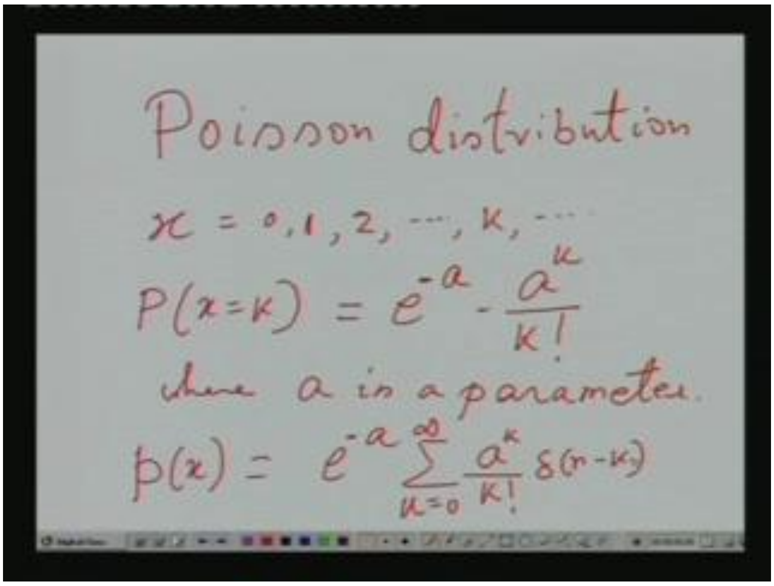

Here, x it takes values like you know 1, 1, 2 dot dot dot dot k dot dot dot dot. The probability of x taking k, will be given as e to the power minus a into with into a to the power k by factorial K, where a is a parameter; question is can a be negative. You see a cannot be negative is, because if you consider, this in a negative power by odd powers of k odd values of k that is odd powers of a here. Say a to the power 1 or a to the power 3 or a to the power five and all than; obviously, if a is negative the entire thing becomes negative right. And probability cannot be negative.

So, from the in the definition itself, it is implied that is a positive number. And depending on the a, that we choose you get 1 particular distribution or other right. So, if that be. So, these are corresponding probability density function. So, this is just a probability, it is not even probability distribution, it is just the probability of x taking k. So, corresponding density function, as we discussed the other they will be what they will be impulses at 0 at, 1 at, 2 at k and like that. And each impulse a that impulse at k'th position we will have a weight given by this.

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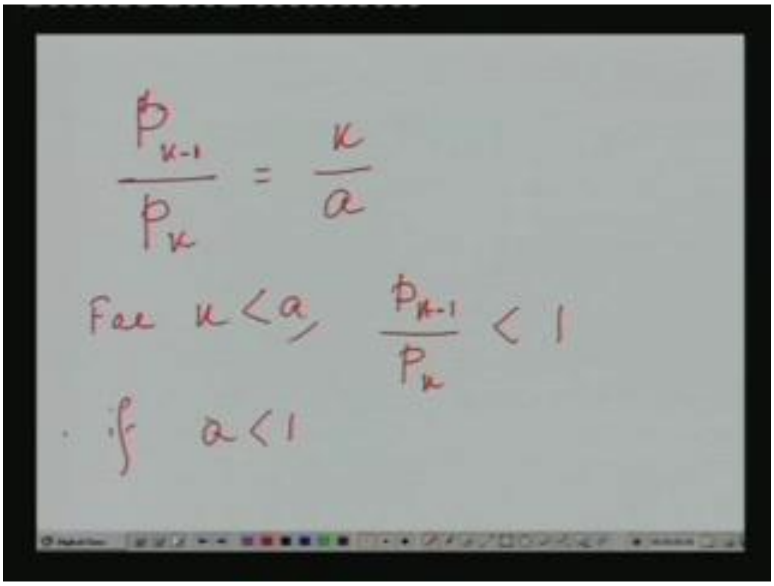

This is the probability density function for these all right 1 thing you see, for this distribution, if we find out this ratio, where P subscript k is nothing but , the probability that of x taking the value k. That is in an expanded form we wrote earlier it like: that P within bracket x equal to k in short, we are writing it as P subscript k. So, if you take the ratio of these 2 that factor a to the power minus alpha cancels. And you now after some cancellation and all that you can easily verify that you get things like this. Now, a is a positive number. Suppose, our a could be integer or a could be you know fraction all right. Now, as long as k remains less than a, suppose k starts at 0 then 1 then 2 then 3 like that it approaches a. As long as k remains less than a this ratio is less than 1.

So; that means, for k less than a this is less than 1, but since k is increasing, what is happening is this. I mean the 2 values Pk minus 1 and Pk they are coming close, because initially it was 1 by a then 2 by a 3 by a. So, this is increasing, which means; the gap between the 2 is decreasing. Finally at k equal to a if a is an integer or k equal to you know the nearest integer from below from a actually.

I mean if you consider a to be a fraction you will just consider, the integer just below a. When k is equal to that then it reaches, it is value it is maximum value actually in that in the sense after that this will become in the this will go on increasing. So, at that point the gap between these 2 is minimum. Other then when k crosses a and goes beyond then what happens is that you know these becomes grater than 1 progressively which means; Pk then decreases.

So, initially Pk goes on increases; increasing around a it reaches its maximum that is the maximum I was taking of then as k goes beyond a far and far you know from a Pk again decreases. Say around a it has it is maximum right. From this you can also see that, if a if a is less than 1 then; obviously, from this discussion, it follows that Pk has, it is maximum value only at k equal to 0, because P as you go for P1 or P2 you are crossing a.

So, the value Pk starts decreasing which means Pk has it is maximum value at k equal to 0; otherwise as I told you earlier Pk increases and when you reach the vicinity of a it reaches, it is maximum and then it starts falling again right. So, actually if you want to plot.

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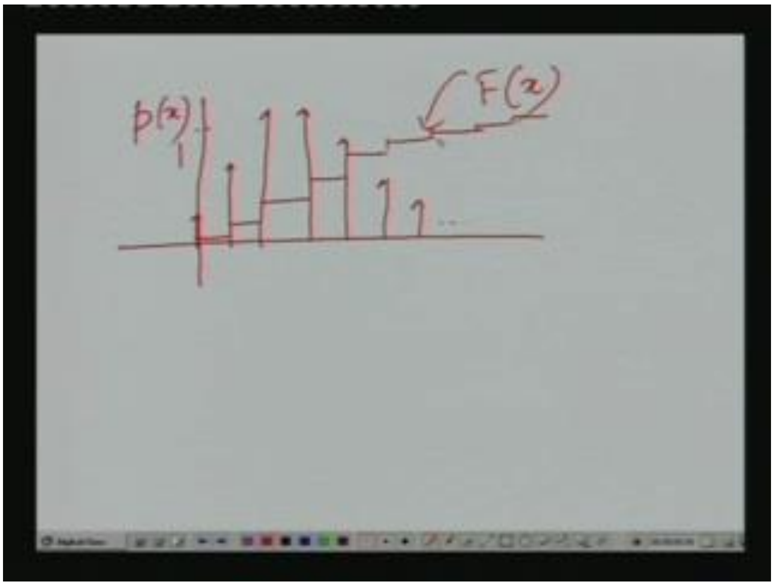

It will be like this impulses; it should be Px rather. So, let me. So, these are the impulses and Px is the summation of the these impulses actually superimposition of this impulses. And distribution function will be like this. You see the gap is reducing since, Pk is no on longer increasing you know it saturates, because like this it approaches 1. This curve is your Fx this for Poisson distribution.

Now, we have considered binomial distribution, we have consider of course, Gaussian distribution, we have considered uniform distribution, and we had considered Poisson distribution, but, these are not all there are other distributions also you know for instance airline distributi on or Rayleigh distribution and all that. So, is not possible to cover all of them they have got in this course the have got the standard expressions and all that.

So, I would recommend that you consult the book by Papoulis or some other book. And then just receive what those expressions are an all that. I now move to what is called conditional distributions and conditional densities. Say when I discussing, just probabilities then after introduce in the notion of probability you now I just move to something call conditional probability. And using conditional probability I finally, arrived at what is call total probability and based theorem.

Now, we are no longer discussion this probability of a particular event. Rather we have discussing the probability distribution of random variable. With the with a special type of probability. If it in this context 2 we would like to see, what is mean by or whether we can construct similar, notion like: conditional probability distribution and conditional probability density function.

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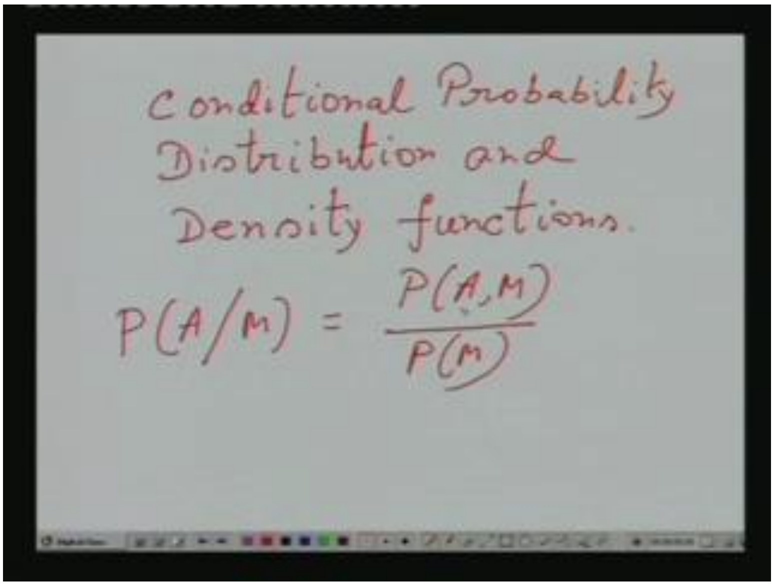

Now, we now just you recall earlier, you have seen that P of some event, A condition to the fact that a event M is to take place. Then the probability of the event, A is occurrence that was denoted in like this that was called of conditional probability of A subject to the fact that AM as already occurred that was define this, where P of A coma M means the joint probability of the events A and M taking place simultaneously, right and P of M is the just the probability of the event M.

That is how it was defined. We can use this definition to obtain similar notion for conditional probability distribution function and formed that will go to conditional probability density.

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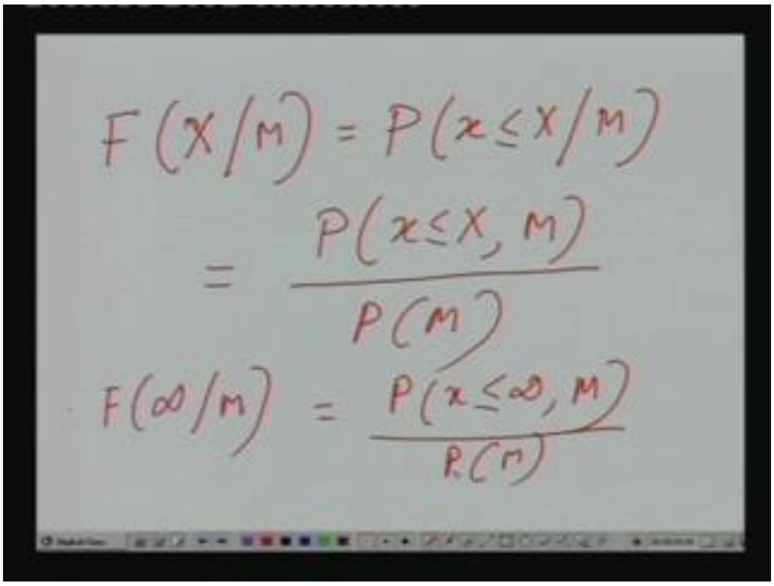

Now, here we will consider this thing you now; X is some value we are taking that to be some real number this is nothing but , probability of is random variable x taking value less than equal to X subject to the fact that the event M has taking place. So, earlier we are considering just conditional distribution and that time we have F of X was equal to what. That is the probability of the random variable x taking values less than equal to X mind you I mean this set the random variable x generates events like this.

That x less than equal to X means; what it actually amounts to a collection of events for which, the random variables take values less than equal to X. So, it is a probability of those that event subject to the fact that the event team has taken place. That is precise mathematical meaning of this. Now, here I can simply use the notion of conditional probability as done before and this will be nothing but . So, this is this a joint probability and divide by P of M. This is a joint probability divide by P of M.

So, this is we expressed what is mean by conditional distribution function. Now, you now after all from this definition, it is should be clear to you that this distribution function also satisfies all those properties, which a simple distribution function probability distribution function was satisfying. Just to explain that first consider, that the property first consider this thing F infinity by M that means this equal to P.

Now as I told you earlier also by our construction by order of definition of a random variable X less than equal to infinity. Now there cannot be any event for which the random variable can take value infinity. So; that means, if you consider, the event for which x is less than equal to infinity then I have to consider all possible events, because for all of them the value of x is finite. Which means; x less than equal to infinity is a total set for total event.

So, joint probability between the I mean total event and just an event M will nothing but , since event M is substitute of this. This will be nothing but , P of M only which means; this is equal to P of M dived by P of M of course, which is equal to 1. So, that property number 1 is satisfied. In a similar manner in a we have considered. We now consider, property number 2.

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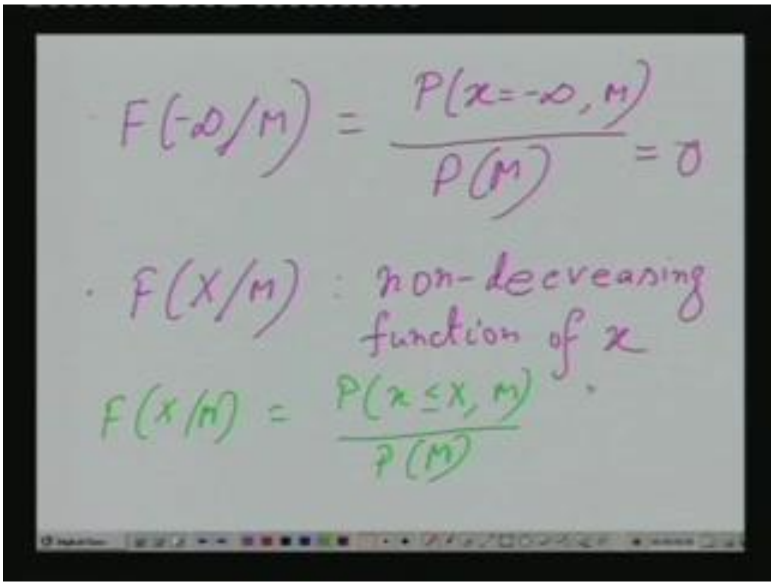

That is F minus infinity subject to the M again this is nothing but , since there is nothing left of minus infinity as a told you last time this actually amongst to this, but, I as a told you last time also by our very definition; there is no event for which the corresponding value of x can be minus infinity. The corresponding probability is 0. So, there is an impossibly event. So, intersection between the impossible event and M will be the impossible event. For which the corresponding probability is 0.

So, this will be equal to 0 by PM which is nothing but , 0. So, again probability I mean these a property 2 which is satisfy ar. Then third thing this is out change it rather. This is a non decreasing function; obviously, because we know that F x by M is nothing but . And this as this is a probability of course, this is a joint probability, but, in a M is a specified event.

So; obviously, x X; I mean the event if you if you consider x less than equal to is say X 1. You get F X1 by M and you considered x less than equal to x2 you get F x2 coma M an if x1 is less than x 2; that means, this event this is subset of this. This event is subset of the event x less than equal to x2. Which means this probability is less than equal to they are probability Px grater than equal to x2 coma M. I mean let me, write it elaborately for your sake.

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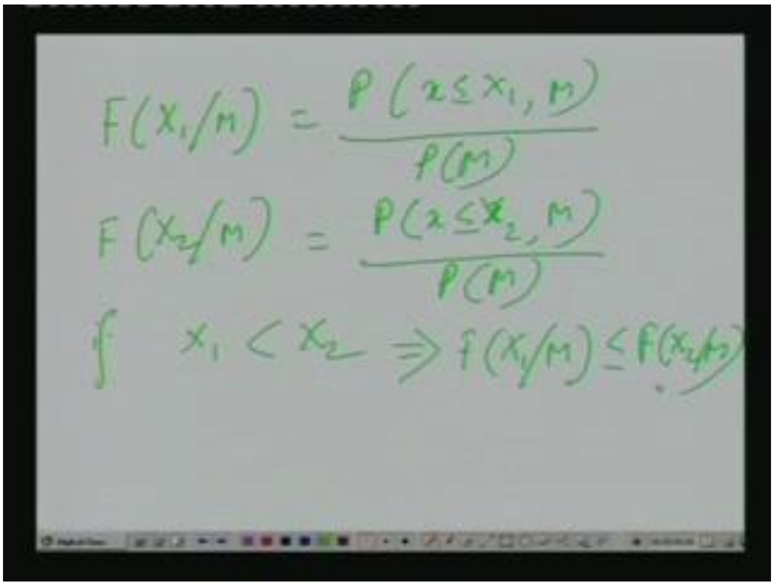

That is, if you write like this and if it is given if X1 is less than X2; obviously, this event is a subset of this event. So, corresponding probability even, if it is join probability corresponding probability or this event is subset of this event you can you will say. That this joint event is a subset this either equal to this or a subset of this. Which means the probability this probability cannot be higher than this. So, this is less than equal to this that is this means.

So, it is as we put a non degrees in function in x; all right other properties also follow very easily.

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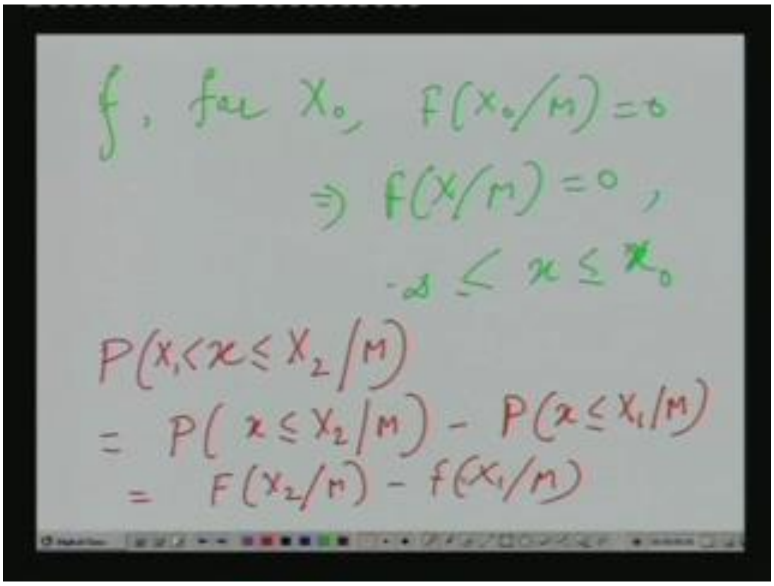

That is if for sum x2 for sum X naught unit is give that F x naught by M is equal to 0, then obviously, for x less than equal to x naught. This follows you know from that non decreasing nature of this function F. Since, and minus infinity its value is 0 and that x 0 is value is 0. And the function can only be non decreasing you cannot fall down. So, all the values, in the range minus infinity x 0 as to be 0.

This is pretty simple the other thing, which is very useful always whether it is probability distribution function or conditional probability distribution function is this. That is what is the probability of x subject to M, but, this is you know you can always write like this. So, it is actually a difference between the 2 distribution functions. This is pretty simple this very much like a you now, this probability distribution function that we consider, last time you can see that whatever, properties we had for the condition for the condition for the probability distribution function they are all valued for conditional distribution function.

So, it is similar manner like, we defined probability density function from probability distribution. We can also defined a conditional probability density function from the conditional probability density we distribution.

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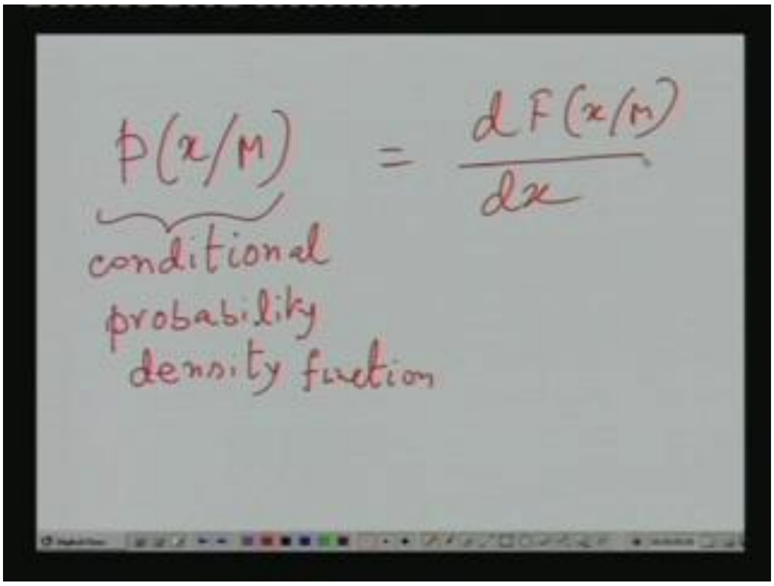

And this we write like this P of this is the conditional probability density. This is nothing but . So, this are derivative. Now, I take an example, where I will be trying to obtain the conditional probability distribution and from that conditional probability density. This example is like this: now I mean that tossing a dice you know it is a fair dice there are 6 faces; 1 face mark to with 1 another 2 then 3, 4, 5, 6, 6, 6 faces. Point is as you dice you now as you toss a dice you now you get a face and the face is marked.

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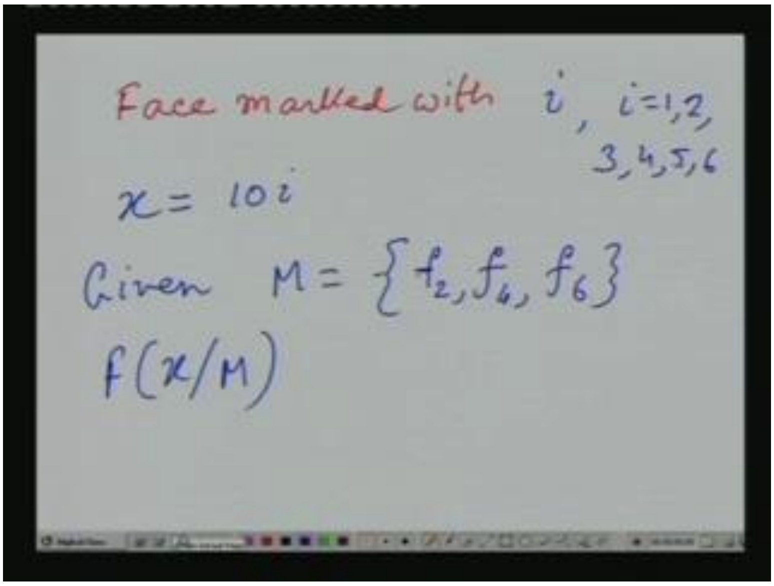

Face marked with I, where I is either 1 or 2, 3, 4, 5, 6, 6 faces. And the random variable x it takes the value 10 I this is. So, if the face 1 turns up it takes the value 10 if 2 turns up it takes the value 20 and likewise you can go up to 60. Now, our problem is given M is either face number 2 or face number 4 or face number 6. That is giving that only even number faces are showing up. Subject to this, what is the probability. That is condition to this what is the probability distribution of this random variable x. There is the problem that is we have to find out where M is this all right. So, you know let us, proceed like this.

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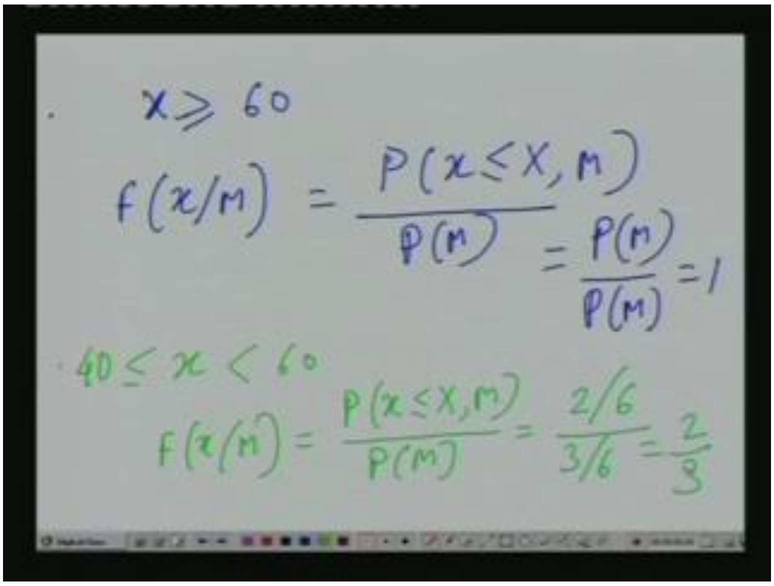

Suppose, we first consider this case x greater than equal to capitals or greater than equal to 60 X greater than equal to 60 that is case 1. So, you now this can happen, when either face 1 turns up or face 2 turns up or face 3 turns up; up to face 6 tuns up, because you are there get 10, 20, 30, 40, 50, 60. If it if X is 70 or 80 or even 60. It means; that any event taking place, will satisfied this side, but, you now we are condition to the fact that not any event, but, only even faced. Even numbered faces are turning up.

So, in this case. What do we have this consider. Firstly, what is P of M P of M means; the probability of any even numbered face studying up. So, there are 3 even number faces 2 4 and 6 and it is a faired dice it is not biased. So, P of M will be what total number faces was 6 out which 3 qualify for this. So, probability of that will be 3 by 6 pretty obvious.

Consider this here, in this case X could be either equal to 60 or 61, 62, 63 anything you now any number above that. Now, what is joint probability of this happening you see, we have to see the intersection of the 2 events right. That is X lying either at 60 or above and then even number face turning up I mean even numbered face turning means; either you get 20 or 40 or 60 for each of them.

The corresponding x is indeed less than X, because if it is if the event if a face number 2 arrives then x to x 20 that satisfies this; I mean that is part of this, because 20 is less than equal to X for X is above 60. Or if it is 40, then again this is part of this, because x equal to if suppose, face number 4 turns up than x is 40 and that satisfies this. So, that included this including this, because 40 is less than is equal to X where X is greater than equal to 60. And say applies for face number 6, because 60 less than equal to X where, X is greater than equal to 60.

So, intersection of the 2 events is nothing but the event M itself. So, this is nothing but P of M divided by P of M which is equal to 1. This is case number. So, when x, because of equal to 60 or exceed that the corresponding conditional probability will be equal to 1. Then considered, the next range less than 60 and greater than equal to 40. In this case. Now let us consider, the intersection here, M means; either face number 2 or 4or 6. Now, if 2 comes X is 20; if 4 comes X is 40, but, x is lie in this range x can be 40 or 41 42 up to 59.

So, face number 6 has no intersection with this whereas, face number 2 and face number 4 they have; they are part this, because face number 2 means x taking value 20. And the X is in this range between 42 to 59. So, 29 is less than that. So; that means, 20 is contending this same applies to 40 a face number 4 turns up, the corresponding of value is 40 , but, 40 is surely less than equal to X if X belongs to this range.

So, face number 4 that is also part of this set, because a corresponding value of x lies is in this range, but, not face number 6. So; that means, it is nothing but the probability for F2 and F4, the probability of F2 and F4. Now, what is the probability for F2 and F4 out of 6faces; I am the now concerned only 2. And it is a fair dice. So, this will be 2 by 6 and here it is 3 by 6. So, it is 2 by 3.

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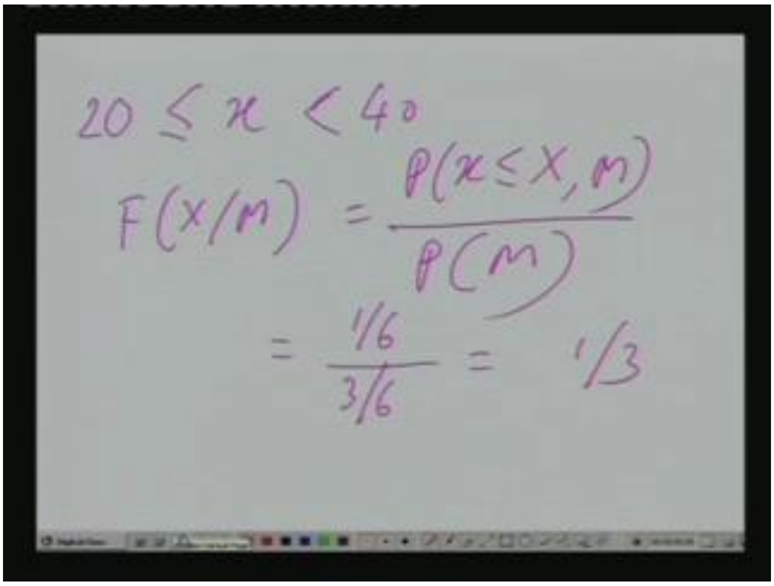

Next, consider this range less than 40 in this case. In this case you have this again consider the intersection. Face number 2, is, because for face number 2 x takes the value 20 and 20 is less than is equal to X when X is a either 20 or 21 on goes up to 39. So, that is satisfied. So, face number 2 is, I mean M and this event M this event they intersect at face number 2, but, not clearly at face number 4 or face number 6. So, it will just be the probability of face number 2 turning up which is nothing but 1 by 6, because it will have 1 face I am now considering. So, this will be 1 by 6 on top and again this is 3 by 6 below which is equal to 1 by 3.

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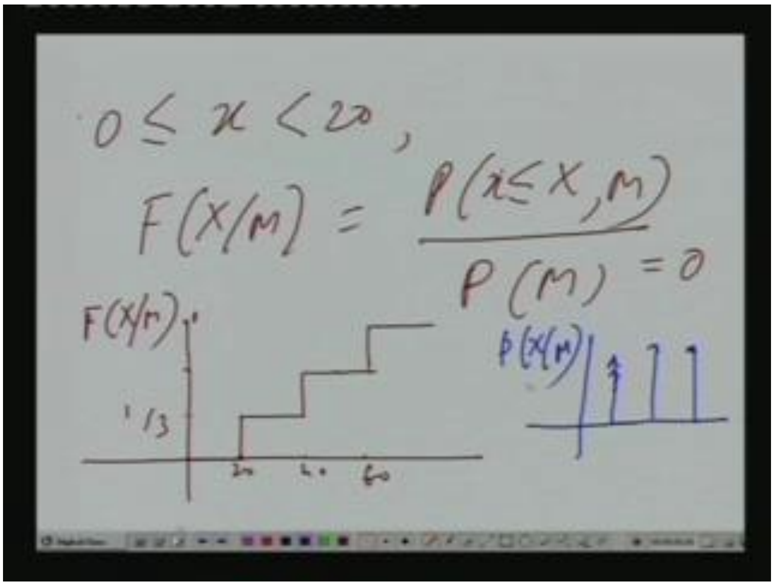

If x is less than 20 and greater than equal to 0; obviously, you now there is no intersection between M and the other 1 that is, if X is in this range then no face where M and this they intersect at 5. There is no intersection, because you started face number 1. So, get 10. Or may be face number 1 part of it. You see no face number 1 is not part, because M corresponds to the events, what event that only even number face is turning up.

So, either 2 or 4 or 6 out of which 2 is ruled out, because for 2 x will be 20 that is out of this range 40 and 60 there also out this range. So, intersection is empty set. And for which that is intersection is an impossible event for which, the corresponding probability is 0. So, this will be 0. So, if we really plot this; you get like this: function like this you know. I think this is 1 by 3, this is 2 by 3 and here it approaches 1 like this. What will be the corresponding density; obviously, the derivative of this.

So, density function will be like, this there is an impulse here, there is an impulse here and there is an impulse here that is all there is a corresponding density all right. Consider, another some other example, you can call them example or you are call the call them results also; because these are very useful in practical applications. So, I will consider here 2 such examples, which are you now I mean part of theory also. Again we will be you try to try to find out the corresponding conditional probability distribution function given a particular problem. Problem is simple there is a random variable x, which is continuous random variable in our case as you have been seeing.

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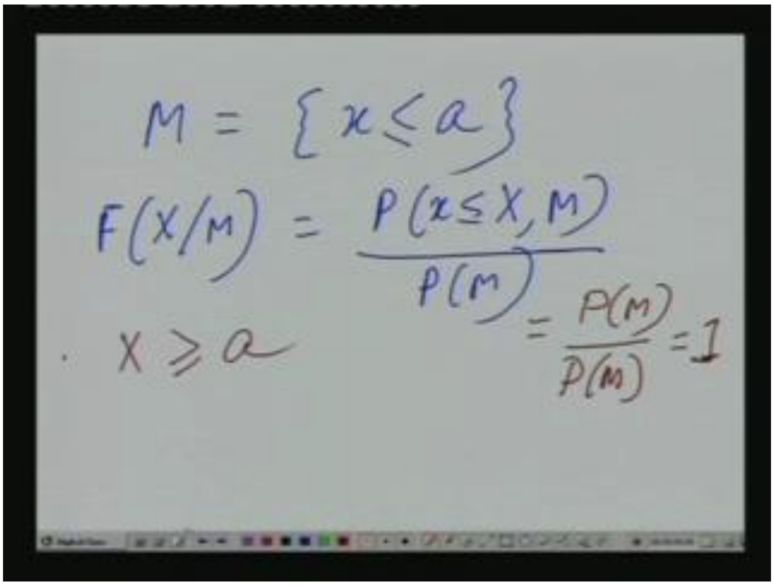

And this given that M is nothing but x taking values less than equal to some a; a is some even number. So, this is the thing that event M means x taking values which are less than equal to a. We have to find out that is as before all right. So, first consider the case with X is greater than equal to a. If X is greater than equal to a then what is the intersection between this and this.

Now, this means; I will consider all values of x for which of the all those x, for which is value is less than equal to a, but if X is greater than equal to a naturally whole of M satisfies this, because M corresponds to only those value I mean those x for which the value of x is less than equal to a. And since, X is less than equal to a greater than equal to a; that means, x is than less than equal to X satisfied. So, in this case the intersection between this event and this event will be the event M itself. So, this will be equal to P of M by P of M which is equal to 1.

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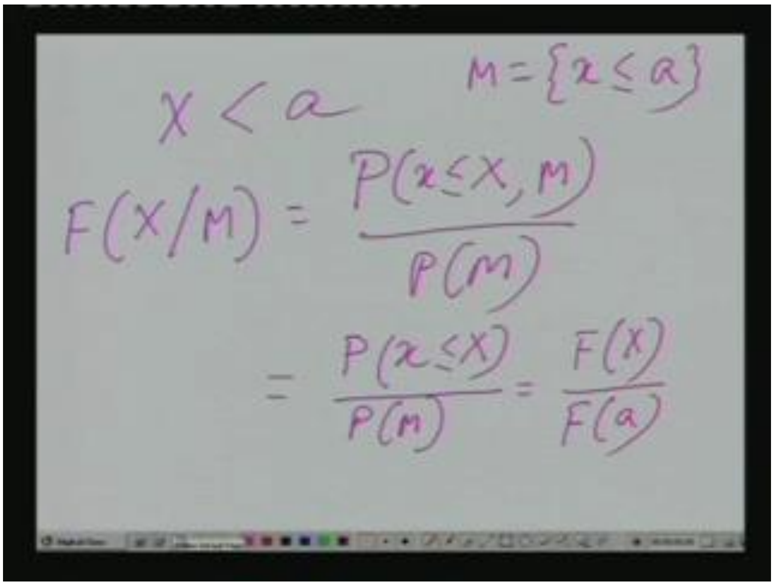

On the other hand, if you now considered, the other case that is X less than a then again. Now, here we have to find out the intersection. Now M was correspond to what M I write separately here just for our convenience this case, but now X is less than a. So, what is the intersection between the 2 since, X is less than a and x less than equal to x, that means; this will be part of this, because this event means; x is less than equal to a and this event means x is less than a.

So, this corresponds to intersection will be what this. X is less than equal to x, X. So, this will be what, but, what is this is nothing but the probability distribution for x and what is this PM means; P of this event x less than equal to a which means F a. This is the case when x is less than a all right, how about the corresponding densities.

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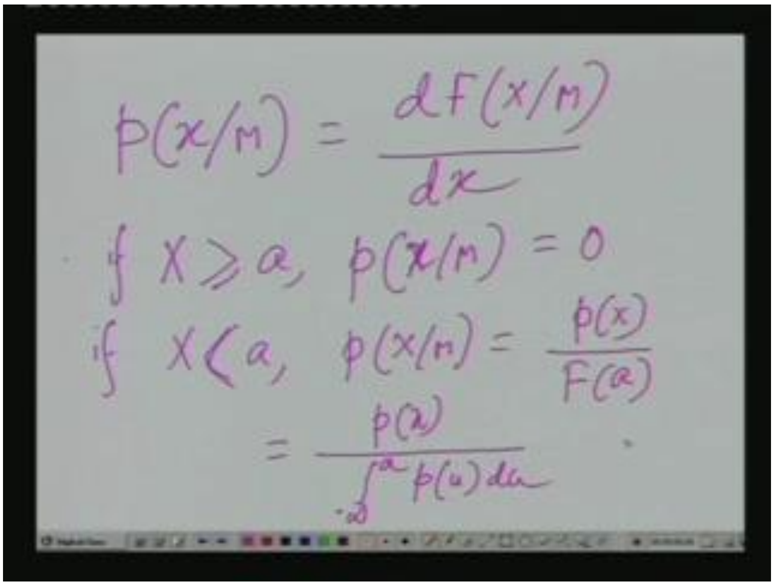

How about this is equal to as we know… Now, in 1 case we have found out that is if this was equal to 1 which means; then the probability density function is 0. What will be this, because you have already evaluate, the expression F of X by M is nothing but F of X by F of M. An if you derived it then you get Px by F M which is nothing but it is not F M actually it should be Fa. Which is nothing but P you can put any value will u du all right.

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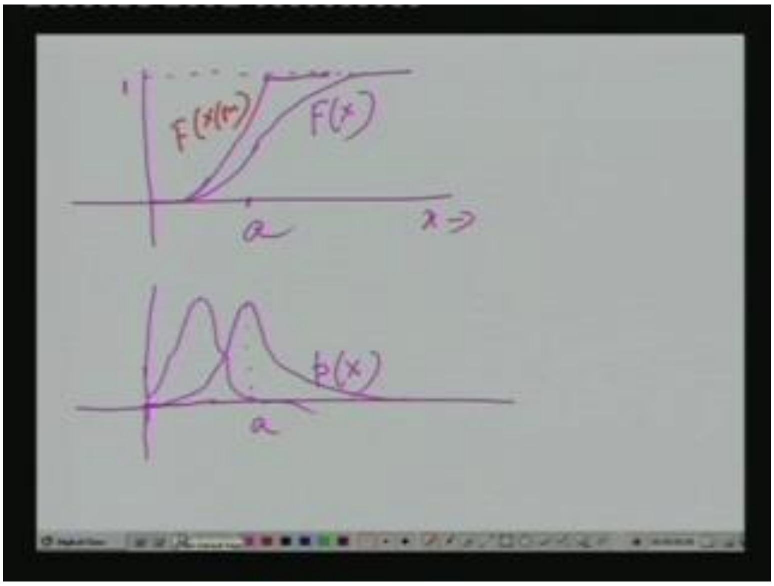

So, if original Fx was giving like this, it was going like this and then approaching 1 and your a is here. Then the conditional probability will be of course, it will go up as you have seen, because till it reaches a it is values F of x by Falselist. And F of x is a non creasing function in x this is x axis. So, this will be only go up and finally, it will approach 1 and it will remains here. So, this is for let mean; erase it here, because there are 2 plot this is for Fx and this 1 is for the event M has been defined earlier.

So, the corresponding density functions will be. If you considered, F of X here it is derivative will be the probability density function. And that will be after this that will be 0 and to the left of the 0. The slope is maximum here. So, it will be you now, reaching it is peak around a, it will go like actually its Gaussian curve, now it is it will be like this. This is just P of X, but how about P of X by M. You have seen that after a we know after a it was going down right after a it was becoming 0. So, it will have some value which will finally, approach 0 here after a. It will have sum value some value, but, after that it will become 0.

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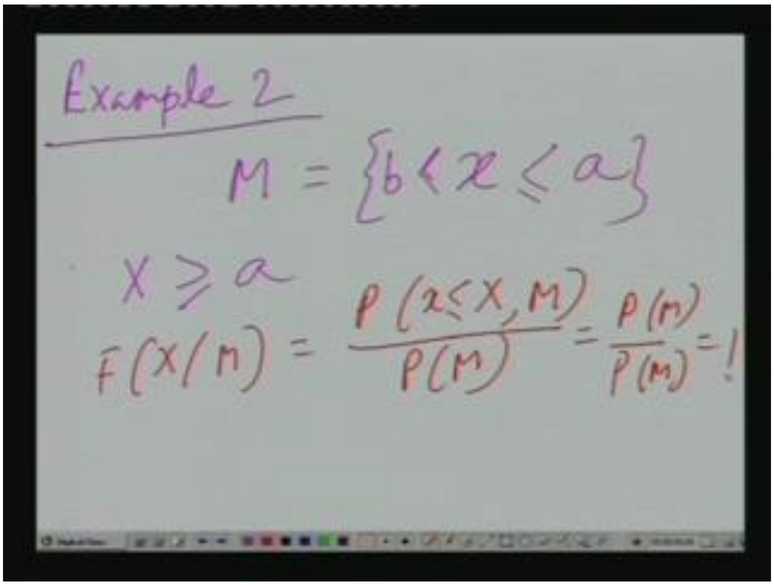

Consider 1 more example, is a more generalize version of this you now here, we have consider the 1 range that is a we just took a number a and the event M was given is like this; that x taking values less than equal to a. So, that was a condition, but, now we want to consider, finite range giving by 2 numbers a and b at the event M will be like this x taking values less than equal to a, but greater than b. This example 2 you may say, we may say x taking values like this.

So, first considered, X greater than equal to a clearly; what is the intersection of this set. Since X is greater that equal to a we know x belonging to this means x belonging here will definitely satisfy this; there will be definitely will less than equal to x, which is nothing but P of M and by P of M that is equal to 1.

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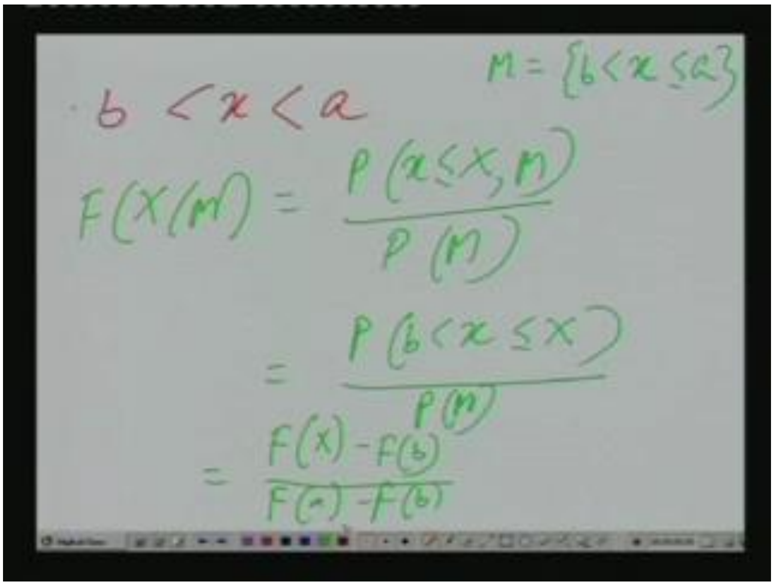

Then considered the next range, that is x less than a greater than equal to b for this. Now for this what is the intersection, because x is now lying in this range X is lying this whereas, M corresponds to what we write separately M x taking values between a and greater than b. And now X is lying in this range. May be I just modified little bit instead of having this. Let me, erase this. So, X is above b and less than a and x less than equal to that X; that and M the intersect where the intersect here by PM.

And what is this, we have seen this is nothing but FX minus Fb divide by PM. And PM is of course, you now I mean if you see PM here it is nothing but if Fa minus Fb ar, and in the other case other branch…

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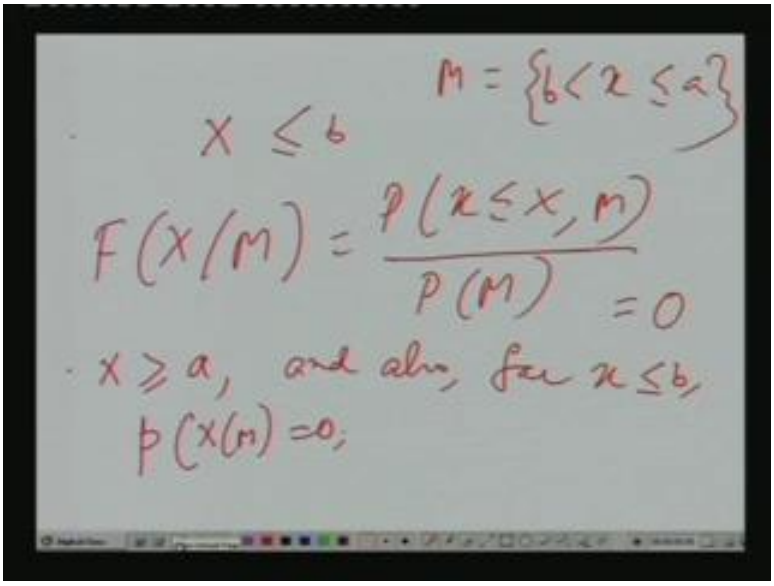

Now, x less than equal to b in that case, what is happening our M is given as this, and what is the intersection here, X is less than equal to be b. Whereas, M corresponds to those x, which line this range greater than b less than equal to a. So, intersection is 5 empty for which probability is 0. So, clearly this will be 0. So, corresponding density will be what for in this probability distribution function is 1 that will be 0 when this is 0 that will be 0. And in between it will be just derivative of this. And then derivative will be what that is for X greater equal to a and also for x less than equal to b P of X by M 0.

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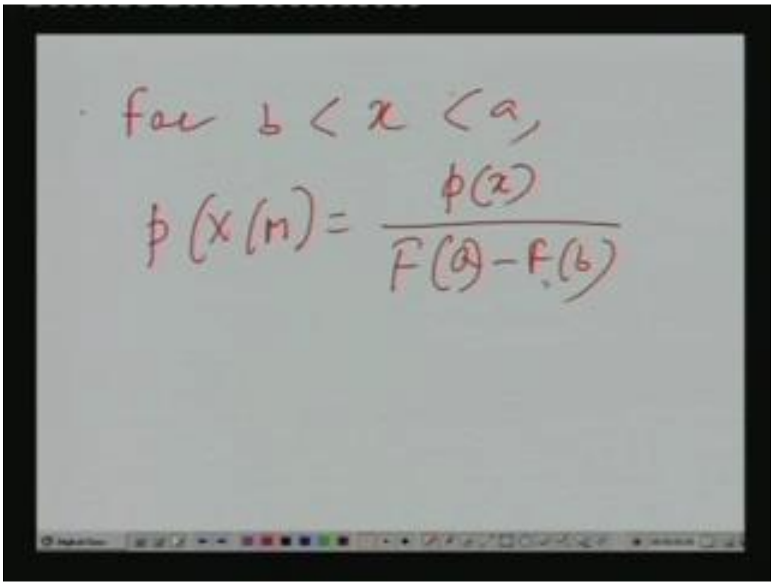

And for x greater than b less than a P of x by M will be P of x, because a deriving it was Fx minus Fb, if you derived we get Px only and the denominator Fa minus Fb. So, that is for today and we have covered enough of this topic on random variable and probability density and distribution function. And they are properties then conditional probability density conditional distribution you have just to considered what is called total probability distribution, total probability distribution and all that will take of some few examples, in the next class.

And then will you proceed to what is call function of random variable. So, along we have been dealing with only random variable then we have to consider, function of random variable like you know I mean if x is random variable then I needs probability density is given, but, then x square also is random variable. What will be is probability density function and likewise. So, that will be done in the next class. Thank you very much.

Preview of the next lecture.

Function of a Random Variable. So, in the last class, we have discussed you know topics up to conditional distribution and conditional density function. In this class, we will considered what is called total probability and base theorem after that may be of an example, will taken up a problem. And then I will switch over to another topic and new topic, which is called function of a random variable will start that topic today.

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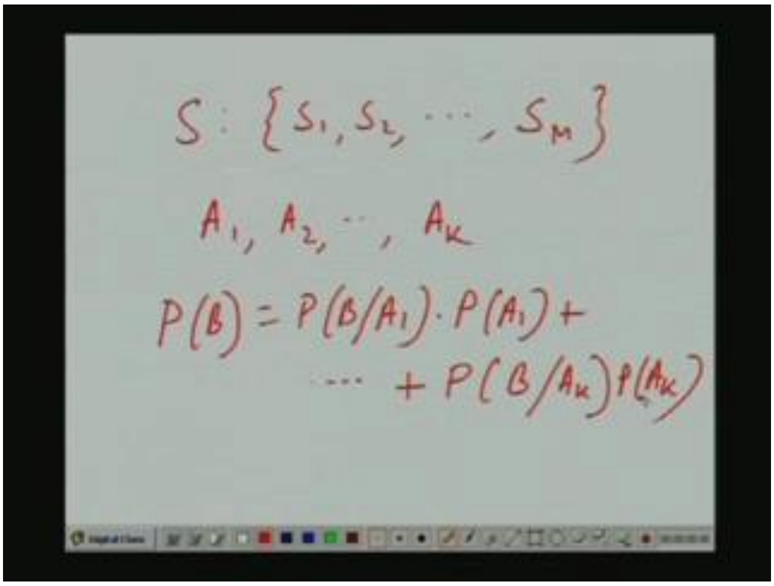

Suppose, S stands for the set of wall possible outcomes and outcomes are like this: S1, S2, dot dot dot say Sm there are m outcomes. Now, if we form we already know that I mean any subset of these continuous event suppose, we considers of events A1, A2, dot dot dots say Ak which forms part partition of S that is A1, A2 and Ak there are mutually disjoint and they are union is equal to S.

In that case as, we have seen earlier that for any arbitrary event B. We can always write PB as PB by A1. That is probability of event B condition to the event A1 times PA 1 plus dot dot dot dot that is likewise PB by Ak P Ak right. While there as for just probability discrete probabilities, but, we can use a same concept to derive the, I mean the corresponding case for probability distributions.

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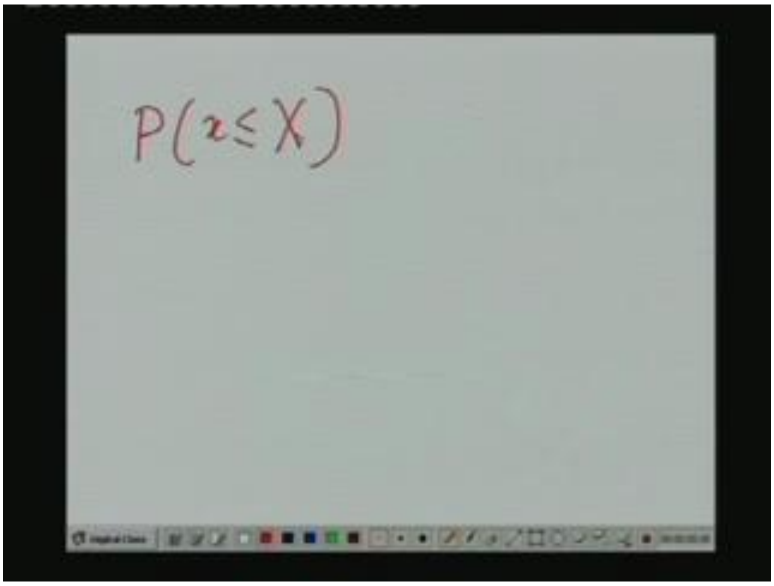

Suppose, it is like this x is the random variable. S as I told you is a set of wall possible outcomes and suppose, there are k partitions A1, A2 up to Ak. Then following the same procedure, we can write then what is this stands for probability of a random variable x taking values less than equal to some giving number X. Actually this x less than equal to X denotes and event; that is, it is a set of wall all those outcomes for which a random variable takes values less than equal to some pre specified value X. In fact, this is what we call as we recall.