Indian Institute of Technology Kanpur

National Programme on Technology Enhanced Learning (NPTEL)

Course Title
Digital Switching

Lecture – 06

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Okay, so in my previous lecture I was talking about the expansion of crossbars using cross bar basically building up a larger switches using multiple of them in multistage interconnection configuration. We figured out that two stages cannot give me strictly non blocking property, but three stages probably can and I will show yes they can and they this requires much less cross point complexity compared to using a single crossbar of the same size okay, we will actually show that.

But before we do that we would like to actually look into how we will estimate the blocking probability of a system. And then of course, they are two kind of blocking estimations blocking probability estimation that we need to also learn to differentiate and we need to also learn how these two are virtually related to each other.

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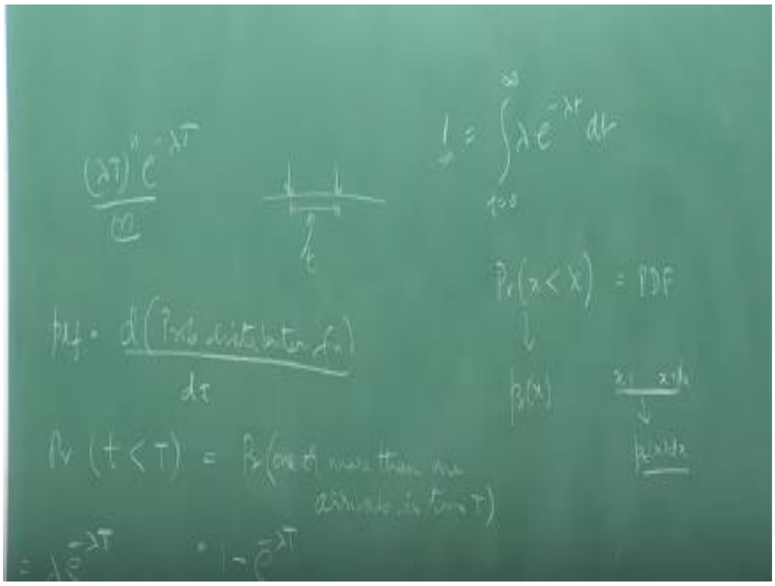

So I will start with a very simple cross bar which is M/N, so remember I am not taking my inputs are not same as the outputs okay. So I am actually having M lines which are coming in and there are n lines which are going out. And they were like the telephone communication which is happening the calls will come and calls will be them connected to some outgoing line.

So if a call comes on this line this will be connected and call is true okay. So the rate at which the calls will come here will be defined using a poison statistics that actually means a point event on a time horizon, I will take time difference time duration of T how many events are going to happen. So if I actually assume the calls do arrive and they are instantaneously completed and instantaneously removed.

So even happens that is it, normally whenever a phone call will come it will be held up for certain time so line will get occupied, new call cannot come, but if that is not happening at what rate the call will be coming when call is not going on. So that we will define using a poison statistics which is a commonly used model. So we define something called λ which is the number of calls per unit time I will call it call arrival rate okay or arrival rate.

And this also being used this technique is also used for packet switching systems also. So in time T λxT calls will be arriving on an average okay. So the probability if this is a person statistics the probability that N calls will be coming will be arriving in time T will be given by, now remember when the call is arriving is being instantaneously served and line is made fries word new calls can come in, because if you occupy the line new call cannot come in actually then I cannot make this estimate.

So $\lambda T^N E^{-\lambda T}/N!$, so this will be the distribution okay. So if you look for the probability of no call being arriving here probability of one call probability of two calls arriving in time T and so on. The total sum of these are all mutually exclusive events so when one call is arriving within time period T two calls will not be arriving when two is arriving one will not be happening.

So these are all mutually exclusive events when I sum up all the probabilities this should be equal to 1 which actually implies that summation of $\lambda T^N E^{-\lambda T}/N!$ this should be equal to 1 which is obvious because, so n goes from 0 to ∞ and this is nothing, but a series expansion for exponential. So this will be written as $E^{-\lambda T}$ which it turnout to be equal to 1. So this is a probability distribution and we call it Poisson distribution actually okay.

So that is what is happening when the line when actually in real life when a call will come people will start talking so there will be some time after which call will get cleared okay. So I define that the call duration is exponentially distributed that is my assumption is exponentially distributed and the mean value mean duration okay for every call is $1/\mu$ okay this also comes actually can be derived from poisonous statistics.

So before I move further I need to clarify this so I had define $\lambda T^n e^{-\lambda t} / n!$ that is a arrival statistics but if I want to measure to Britain between two consecutive arrivals what is the time duration let me find out the statistics for this time okay so I want we can figure out a PDF but for finding out a PDF, PDF is probability density function you can always take probability distribution function and do that derivative of this and if the variable is T it has to be / T it is derivative as to with respect to T.

So I think this is very similar that your x is less than X that gives you the your PDF probability distribution function you take the density you will define something like probability density function, so the variable X is going to be in range $x + x_2x + dx$ within this range the probability that x will be lying within this is always given by px x dx okay that is what the density actually means and when it is a probability distribution function this defines probability that x will be having a value smaller than this X that is a PDF okay.

And this is incremental technically that is why you have to take derivative of this to get this okay when I am actually I can use this same mechanism to estimate the exponential distribution for the time so Poisson distribution is related to exponential distribution in that sense the time between two consecutive arrivals actually is exponentially distributed so this can be estimated by if the time is $t < T$ the inter arrival time is smaller than this is this will happen this probability is nothing but that probability that you have more than one or more one arrivals in time T.

Okay and of course this actually means 1 - there is no arrival in time T okay so this I can use this same expression Poisson thing 1- no arrival when n is 0 this will be $e^{-\lambda t}$ and this is nothing but a distribution function see here so I need to just take the derivative to get the density is the probability that the value of x is lying going to lie between x to nx + dx divided by this differential so this is the probability and so this px x will be given I will be giving you the density function.

So let us take the derivative of this will give me now remember this the random variable here is T so this time so I can actually replace it by a t also and if I take this if this is a PDF the area under this should be equal to 1 which is actually true if I take $\lambda e^{-\lambda t}$ dt now this t cannot be negative this can only be positive so it can take value from 0 to ∞ this will be equal to 1 so this actually can be proven now this time exponential this is the exponential distribution.

So I can represent that how many calls will be served or how many calls will be finished essentially is a service rate the μ number of call swill be serviced per unit time by the line that essentially service rate will be decided by the call duration average call duration so that is also essentially being assumed to be exponentially distributed okay so with this actually now I can analysis this switch and try to estimate what is the blocking probability now before that when the blocking is going to happen.

If my number of outgoing links n is greater than M will there be a blocking whenever a call will arrive call can always be connected to an outgoing link because it is available and when you are talking anyway the call cannot come at your line only on the free lines the call can come but you will always find an outgoing link so there is no blocking here so blocking will always happen if n is going to be < than M so when m is > than or equal to M then blocking will not happen blocking will happen only in this case.

So what is the blocking probability so I have to find out that is all what is the probability that all the lines here are going to be occupied we need to estimate that how to do this so we can actually use here Marko chains the concept of that so the idea is that I will identify the state of this switch if only one line is busy switches in one particular state this line can be anybody it does not matter so for one line is busy switch in state1 when no line is occupied switch is in state 0 and so on I can actually then been switch can be in state n when it is in a state n m is larger than n.

So there are some free lines call can still come in but they cannot go through so switch is in blocked States so probability that this switches in state n will give you the probability of the switch being in blocked state okay this estimate or this number will be actually useful to us at some later point of time then we will actually start doing estimation for a three-stage interconnection so when you are switch so what I will do.

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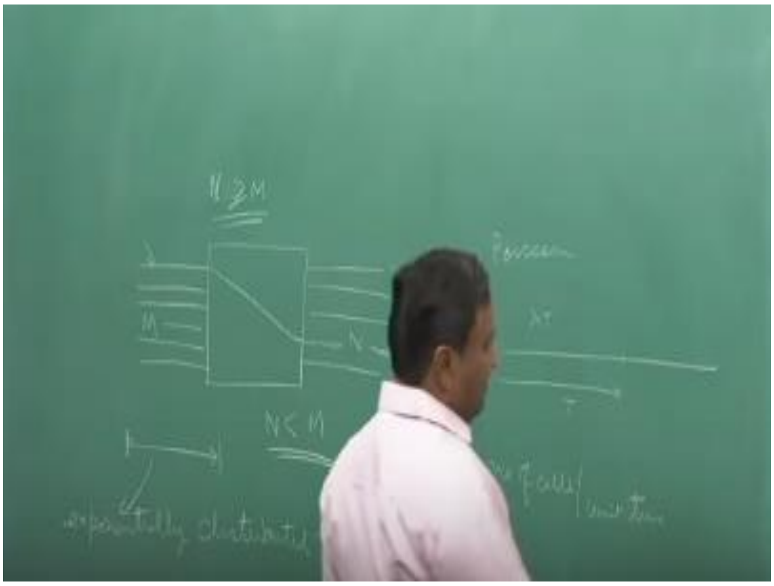

Is I will represent these small round circles will represent States so I am I can actually be instate 0 I can be in state 1 I can been straight to and so on I can be in some state i and I can be in some state n in fact truly speaking if this one is line is occupied. All over all free this is one particular state next line is occupied all others are free is another state but they are equivalent states because there is only one outgoing line which is occupied so I am actually not I have actually have reduced by state space by merging all the equivalent states.

So there are only ten non-equivalent states which are possible in the system so when you are in state 0 there is no call I need to find out at what rate I will be transiting to state 1 so with λ number of calls per unit time the calls are arriving. What is the probability that a call will arrived in any one of these lines.

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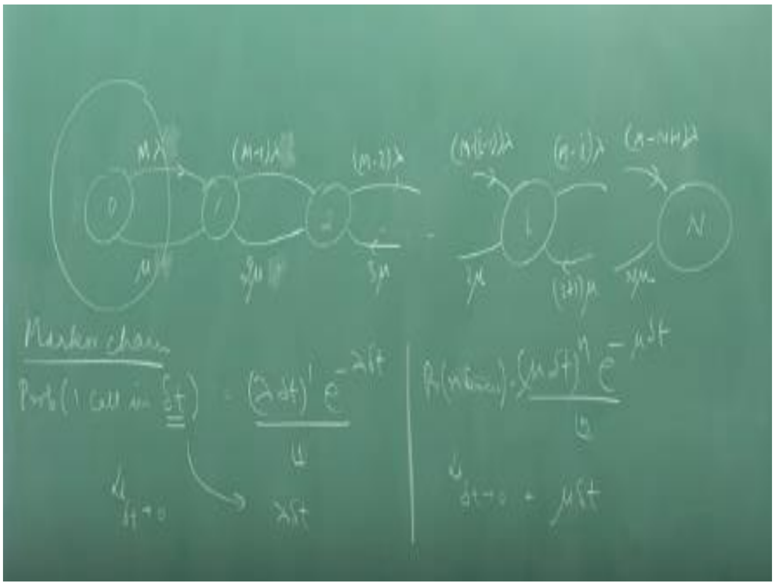

So there is only one line which I will consider probability that one call will be coming in sometime elemental time δT I am going to take it is very small time and as Weill solve we will figure out this δT is immaterial because it cancels out when we will try to estimate the state probabilities so I can use the poison statistics so this will give me λ into δT^1 is found one in fact I can estimate the probability for to caller i will also in time δT but δT I will take so small that probability of having two calls.

Will be still smaller actually so and then of course I will have 1 factorial n in limit when this δT goes to 0 this will turn out to be nothing but λ into δT so that is a probability or that is a instantaneous probability in small time δT that one call will be arriving on one line, there I am such lines so the probability that one call will arrive will be given by m x $\lambda\,\delta$T that is the rate at which the calls will be arriving, so that is a transition rate for going from state zero to state 1, okay. Now when you are in state 1 there is only one call which is going through.

Remember it is like I told you that exponential distribution and Poisson distribution are related to each other, so it is like there was only one call and this call is going to be serviced and there is only one server and the service rate is μ so in time δT μ x δT e^n $e^{-\mu}$ δ T by those many calls will be served so probability that n calls will be served is μ δ T e^n $e^{-\mu}$ δ T/n! since there is only one server only one call can be served so in time δ T when limit when δ T goes to 0.

n service to be given so when I am looking for n = 1 if this becomes μ x δ T that is a service rate. If the two lines which are occupied in time δ T it will be 2 μ δ T that will be the service rate. So I can actually come from one back using μ x δ T when you are in state 1well line is occupied from this side. The free incoming lines are M - 1 so the rate h way at which the calls can arrive is. M - 1 x λ δ T and where you are in state 2 there are two calls which are there.

With a small time δ T the probability of chances this basically is the probability transition probability, okay. In small time δ T remember is a probability expression which have been approximated as λ δ T there is a probability expression, so chances are that it will be 2 x μ x δ T and now important thing is that if I want to if this is a system which is in a steady state condition.

So the probability of being in certain state will not change it will remain it is a constant actually and something can always remain constant if I take any closed surface in this is known as Markov chain this is called Markov chain and this is being used extensively for queuing theory for analyzing the Q Q's performance, so this is also technically a queue, so if I take any closer surface like this.

So the rate at which you are going to move out of the surface and rate at which you are coming into the surface has to be equal if the switch is in steady state, which implies that probability that you are going to be instate 0 is p0 and that is a probability of transitioning going from 0 to 1 so that should be a transition rate now, should be this and this should become p1 $\mu\delta T$. Now interestingly this δT gets cancelled because it is common on both sides you actually make any closed surface here and build up a balanced equation this is known as balance equation.

So the rate at which you will go out of the surface and read rate at which you will come into the surface has to be equal, okay. Because this system is in steady state under that condition δ T will always cancel out, okay. So ultimately we normally in when we build Markov chains we do not write this δ T because of this condition, when I put δ T it is a probability, probability of the transition taking place.

But when I am NOT putting δ T this value can actually be more than one, remember probability cannot take a value more than one that is why δ T was multiplied ideal it should be probability is a transition probability going from one state to another state what is a chance it will happen, so that is what we do in finite state machines, okay. So δT now actually can be removed. So ultimately what how this whole chain will look like, so chain will now look like if you go from here M – 2λ 3 μ.

So I have written these are basically now become transition rates not the transition probabilities actually okay so the probability are proportional to this so proportionality constant δ T is be removed out so balance equations it does not matter actually. So now I can actually use a balanced equation to solve it my idea is that I want to figure out what is the probability of being in state I, I want to estimate that okay so once I know the probability of being in state I, I can find out the probability of being in state n which is the blocking probability of the system.

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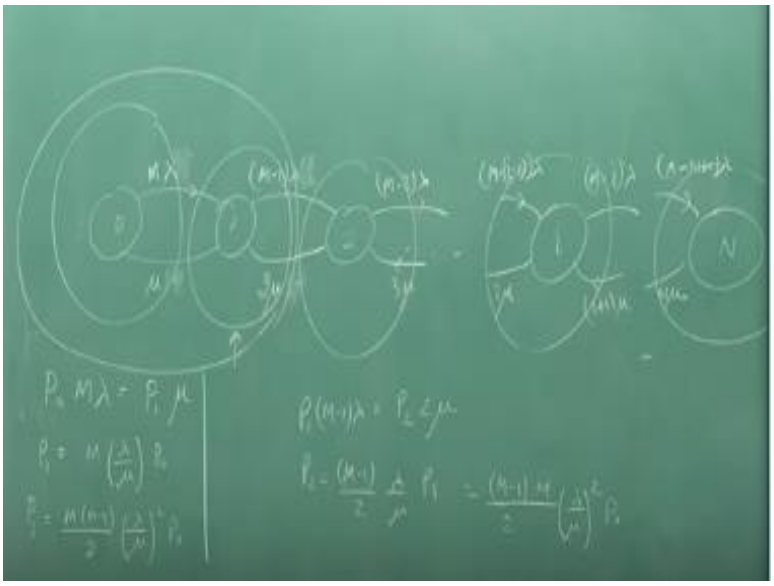

So let us write down the balanced equation I can write down in many ways okay so one of the possible ways is this. So I can actually take a closed surface around every state and then write down the balanced equation so balanced equation around 0 will be p_0 the outgoing rate is mλ this should be equal to incoming rate which is p_1 μ okay so which gives me p_1 as I can solve it m λ by μ p_0 I can look at the second surface and write down my equation this will turn out to be p_0 m λ that is incoming rate one incoming rate is from this side to μP_2 the outgoing rate is μ p 1 + p_1 M - 1 λ okay.

So I just need to have a relation between p_2 and p 0that is what I want I want to represent all the state probabilities in terms of p0 okay once I have that I can use some fundamental axiom of probability to figure out what is p_0 and henceforth I can actually estimate all state probabilities so I need to put out p 1so let me put the value of P_1 there so that I can get p 2so this p0 also I need to move on that side so this will turn out to be m λ p_0 this term comes on the site will become p_0 m λ.

So this cancels with this and this μ will become μ^2 this will be 2 so ultimately you will have alternatively I could actually have taken a surface like this so incoming rate has to be equal to outgoing rate and you would have got this equation directly okay so this would have become a relation between $p_2$and p_1 and p_1 I could have replaced from the previous one to get this particular equation.

So doing it this way for this surface when I am taking to enclosing this P 1 M - 1 λ should be equal to p2 2μ you so which becomes p2 is equal to M - 1 / 2 λ by μ 1 which is I can actually keep on doing it so ultimately I can find out what is the probability of being in state I so this will be probability of being in state I will be given by MM - 1 M - I + 1 okay.

Remember their inner brackets okay and of course I* I – 1* 1 and a hair it will be λ by μ this power I p 0 now whatever what is this expression they think right so this particular expression is nothing but a combinatorial it is MC IM ! m- I! I effect Orion the lower part is I !l and this part is what gives you the numerator here so I can write P I s M commute oriole I λ / μ^{I} into p 0 so for every state the probability now can be represented in terms of p0 .So now let us solve for what the p0 will look like.

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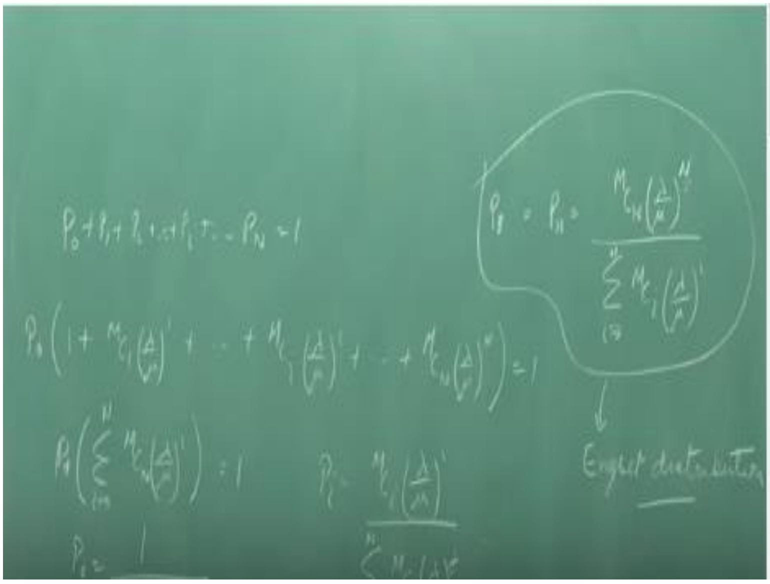

So to find out p 0 remember all the states which always has to be in one of the states so sum of all state probabilities has to be equal to 1 so technically sum of all mutually exclusive events all possible of them and the probability of those events been happening when you sum up all the things together that should be equal to 1 that is a fundamental axiom of probability so I will use that so p0 + p 1 + p 2 and so on + P I and PN should be = 1 so which I can right now p 0 as 1 + MC 1 λ by μ^{1}this should be equal to 1 which gives me nothing but p 0 is this is not a closed form solution if n would have been equal to M then I could have got a closed form solution here I cannot okay.

So p0 into ∑ i going from 0 to n MC n λ / μ^{i} is equal to 1 so p 0 is 1 divided by ∑ i = 0 to n okay so you can now find out what is going to be probability of being in state I mci in fact I should write a different index here so I can put j because I have used I here so I have to use this so this is the probability now what is the probability of being in blocked state so what is the blocked state when the switch is instate n and when I do that probability of switch being in blocked state we call it also as a PB okay.

Probability of being in blocked state that is the way I will be representing this so this will be MC n λ / μ that is power n ∑ i goes from 0 to n but that is a blocking probability for am by n composites which we call it m by n composites which and this also is known as AND set distribution okay.

Acknowledgement

Ministry of Human Resources & Development

Prof. Satyaki Roy

Co-ordinator, NPTEL, IIT Kanpur

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Sanjay Pal

Ashish Singh

Badal Pradhan

Tapobrata Das

Ram Chandra

Dilip Tripathi

Manoj Shrivastava

Padam Shukla

Sanjay Mishra

Shubham Rawat

Shikha Gupta

K. K. Mishra

Aradhana Singh

Sweta

Ashutosh Gairola

Dilip Katiyar

Sharwan

Hari Ram

Bhadra Rao

Puneet Kumar Bajpai

Lalty Dutta

Ajay Kanaujia

Shivendra Kumar Tiwari

an IIT Kanpur Production