Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications
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Lecture – 09
Simplification and Example of Minimum Mean Squared Error MMSE Estimate for Wireless Fading Channel

Hello. Welcome to another module, in this massive open online course on MMSE or Bayesian, MMSE estimation for wireless communication. So, we are looking at the MMSE estimate, of the wireless channel coefficient, of the fading channel coefficient h, based on transmission N pilots symbols.

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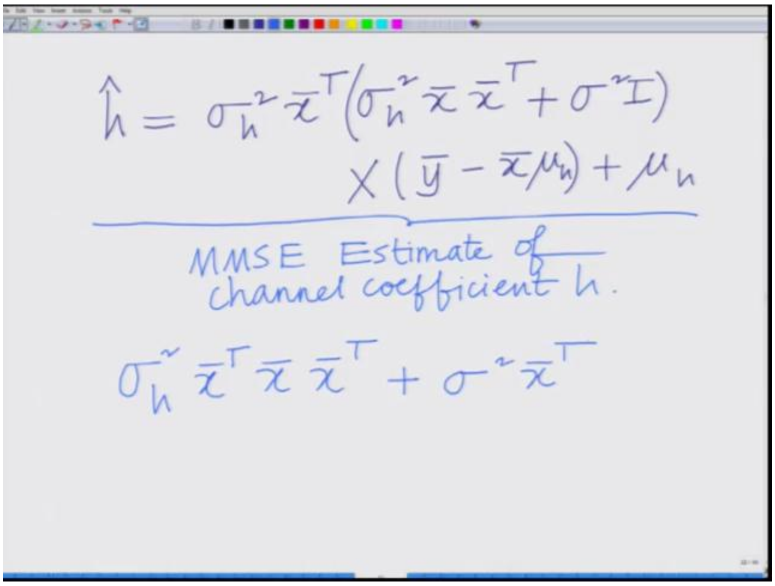

If you denote the pilot by x bar, we have derived the expression, for the fading channel coefficient. And in the previous module, we have shown that, the estimate of the fading channel coefficient is h hat, equals, this is given as well, sigma h square, sigma h square, times, x bar transpose, into, sigma h square x bar, x bar transpose plus, sigma square identity, times, we have y bar, minus x bar h, plus x bar mu h, x bar times mu h, plus mu h. This is the MMSE estimate of the channel coefficient h, correct? This is the MMSE estimate, of the channel coefficient, MMSE estimate of your channel coefficient h.

Now, what we going to do, is we are going to simplify this expression, to obtainable expression, which is much more simpler to analyze and basically, to derive insights from. So, we are going to simplify this expression. Now realize, similar to the wireless sensor networks in scenario, we can part with sigma h square, x bar, transpose, x bar, x bar transpose, plus sigma square, x bar transpose. Now I can expand this quantity.

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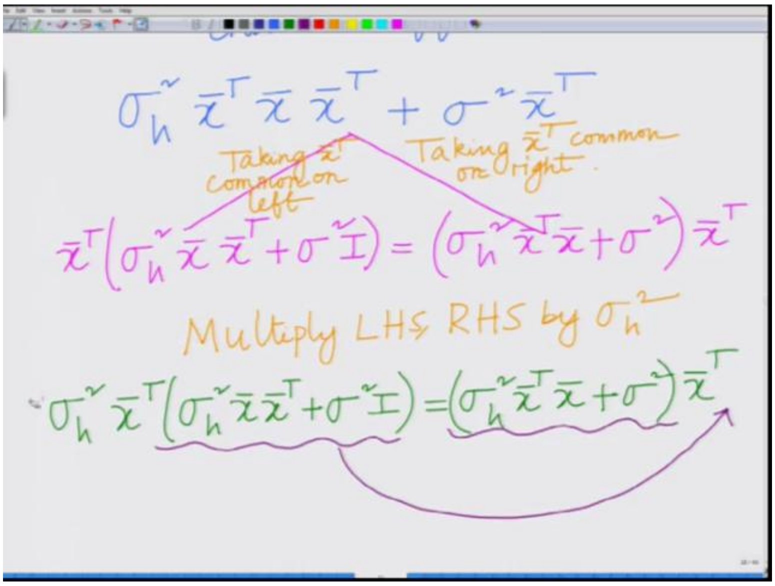

Now this quantity, observe that this quality, I can expand it in 2 ways. I can either take x bar transpose common on the left, or I can take x bar transpose common on the right. So, if I take, if I extract x bar transpose on the left, what I am going to have, is this is x transpose, into sigma h square, x bar, x bar transpose, plus sigma square times identity, and this is, if I take x bar transpose common on the right, then what I going to have is, a basically I am going to have sigma h square, x bar transpose, x bar, plus, sigma square, times, x bar transpose.

And naturally both this quantities are equal, because basically they are simplification of 1 and same thing. In 1 we are taking as x bar transpose, x bar transpose common on the left, in the other, we are taking x bar transpose common in the right. So, these are simplification of the same expression, therefore, these are equal. So, we also write this, mention this, clearly over here. Taking x bar transpose, common on left hand side, taking x bar transpose common, on the right, and these 2 quantities are therefore, equal.

And therefore, what does this implies, this basically implies, now, what I am going to do, multiply both sides by sigma h square, first, multiply lhs and RHS by sigma h square. If you multiply LHS and RHS by sigma h square, we have sigma h square, into x bar transpose, x bar transpose, sigma h square, x bar, x bar transpose, plus sigma square identity, this is equal to, then sigma h square is a scalar, so it can anywhere. Sigma h square, x bar transpose, x bar, plus sigma square, times, x bar transpose. Now what I going to do, is I am going to take the inverse of this quantity, bring this quantity on to the right, I am going to take the inverse to this quantity, sigma h square x transposes x bar, bring this quantity on to the left, over, bring this quantity on to the left, over here.

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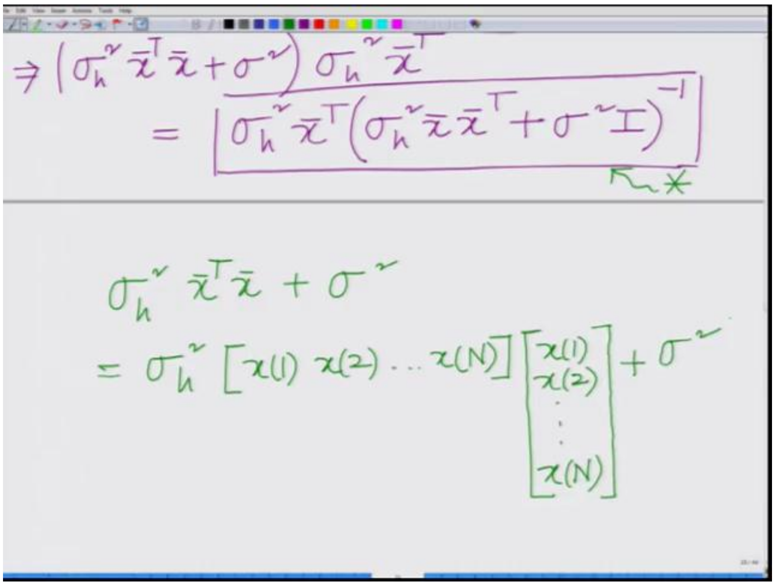

So, now this basically, we can write, this implies, sigma h square, x bar transpose, sigma h square, x bar transpose, x bar, plus sigma h square, inverse times, sigma h square, x bar transpose, x bar transpose. This is equal to, take this inverse on the right, I have, well of course, I have to multiply by sigma h square, by. So, this has to be sigma h square. This should be equal to, sigma h square, x bar transpose times, sigma h square, x bar, x bar transpose, plus sigma square, times, identity inverse. And now what we see is this quantity. If look at this quantity, if you look at this quantity over here, this quantity, and if you call this quantity, as basically, let us say, this quantity you call it as star, and now you can see, this quantity here, is basically the same as star. That is sigma h square, x bar transpose.

Sigma h square, x bar transpose, into sigma h square, x bar, x bar transpose (Refer Time: 07:05) the square identity inverse. Both these quantities are the same. Therefore, we can write, the MMSE estimate h hat can be equivalently written as, h hat. Now before we write the MMSE estimate, now, we will do 1 more simplification. Notice that, this quantity sigma h square x bar transpose x bar plus sigma square, this is the scalar quantity, notice that this is the scalar, correct? In fact, x bar transpose x bar, look at this, this is sigma h square, x bar transpose x bar, plus sigma square, that is equal to, sigma h square, the row vector x1, x2, so on up to xn, times a column vector x1 x2, so on up to xn plus, sigma square.

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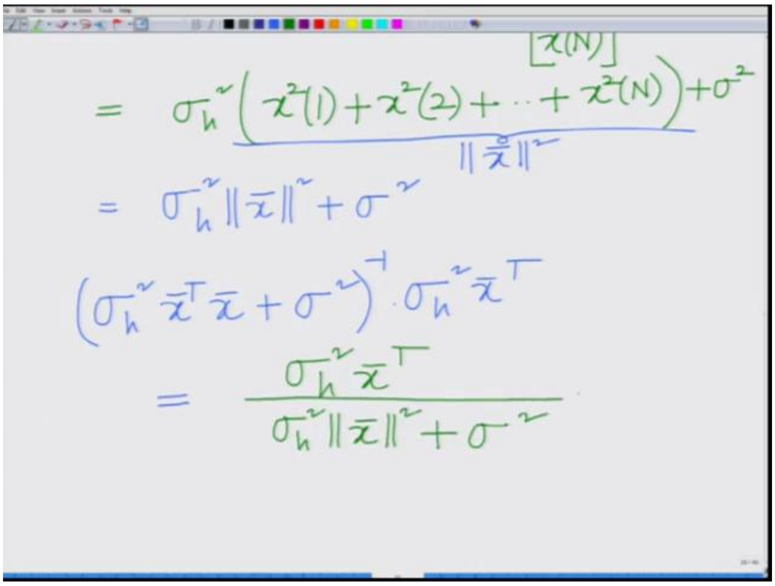

And now if you look at this, this is basically equal to, your sigma h square, times, x square 1, plus x square 2, plus so on up to x square n, plus sigma square, plus sigma square, and if you look at this, this is nothing, but summation x square, x square 1, plus x square 2, plus x square n. This is norm x bar square, correct? This is norm x bar square. So, this quantity is basically your, what is have shown is sigma h square, norm x bar square, plus sigma square. This is the scalar quantity. And there for now, if you take a look at this quantity here, I can write this as, therefore sigma h square, x bar transpose, x bar, plus sigma square, inverse, sigma h square, x bar transpose, equals, now this quantity is a scalar.

So, the inverse is simply going to be reciprocal. So, this quantity is sigma h square, x bar transpose, divided by, sigma x square, norm x bar square, plus sigma square. And therefore, what we are saying is, this quantity is equal to your, sigma h square, x bar, x bar transpose, plus sigma square identity inverse, into sigma h square, into x bar transpose.

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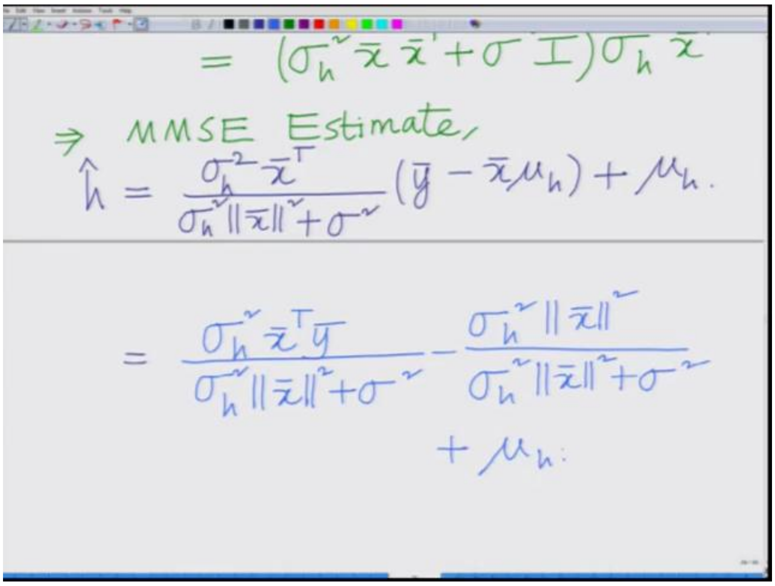

And therefore, the MMSE estimate implies your MMSE estimate therefore, the MMSE estimate we have is, h hat equals, sigma h square x transpose, divided by sigma h square, norm x bar square, plus, sigma square, times, y bar, minus x bar, mu of h plus mu of h. Therefore, the MMSE estimate is, h hat equal sigma h square, x bar transpose, divided by sigma x square, into norm x bar square, plus sigma square, times, y bar, minus x bar, into mu h, plus mu h. This is the simplified.

Now, we are going to simplified further, now we going to simplified further. Now you look at this, I can write this in a form of sigma. I expand this, y bar minus x bar mu h, multiplying by the terms I can write this as a sigma h square, x bar transpose y bar, divided by, sigma h square, norm x bar square, plus, sigma square, minus sigma h square, look at this, x bar transpose, x bar, that is basically your, norm x bar square, divided by sigma h square, norm x bar square, plus sigma square, plus mu h.

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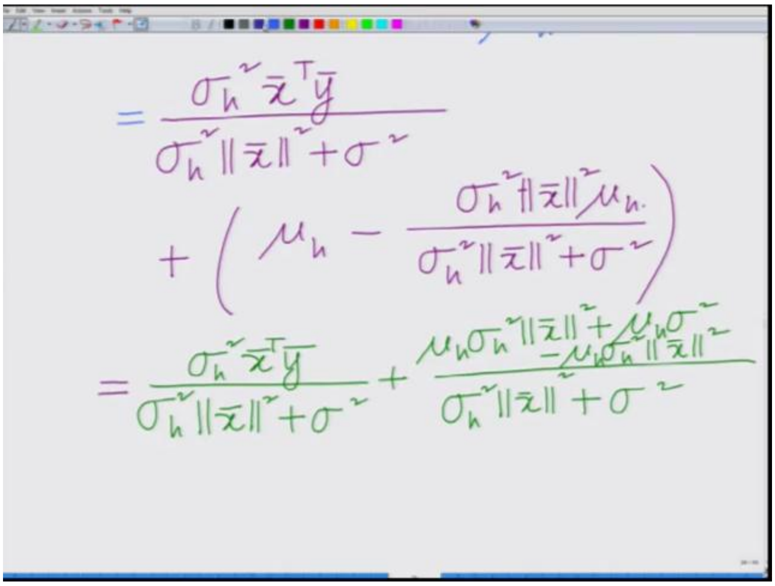

Now, look at this. We have, minus sigma h square, norm x bar square, divided by sigma h square, norm x bar, plus, square plus mu h. Now I can write this as simplified this as, sigma h square, x bar transpose y bar, divided by sigma h square, norm x bar square, plus, sigma square, minus, look at this, sigma h square norm x bar square, norm x bar square, minus, mu h, minus, you will have, well what do you have? You have, sigma h square, of course, there is also norm x bar square, into a mu h over here. So, sigma h square, minus sigma h square, norm x bar square mu h, into mu h, and when you take this mu h, that will give you minus, sigma h square. Let me just write this as a separate term.

So, let me just write this as, let me just simplify this as, plus, mu h, let me just write to explicitly, mu h minus, sigma h square, norm x bar square, plus sigma square, numerator you have sigma h square, norm x bar square into mu h. And now you can see what happens when we simplify, this when we simplify this, what we have is basically, we have sigma h square, x bar transpose, y bar divided by sigma h square, norm x bar square, plus, sigma square, plus, divided by, sigma h square, norm x bar square, plus sigma square, mu h into sigma h square, norm x bar square, plus mu h into sigma square, minus mu h into, sigma h square, norm x bar square of course, these two terms will cancel. And what you have, here is basically net, now once you simplify, bring them in to a common denominator, what you will have is sigma h square, into norm x bar square, plus sigma square, into sigma h square, into x bar transpose, y bar, plus mu h, into sigma square.

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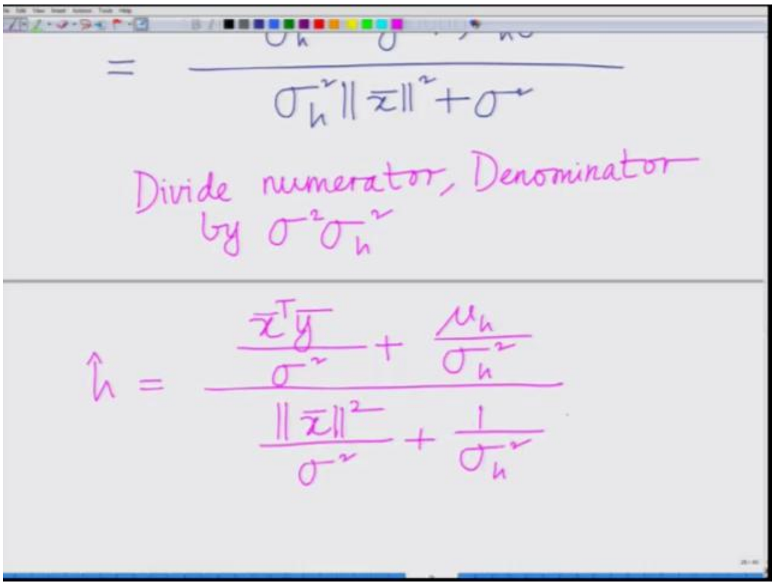

So, this is the simplified expression that you have, sigma h square, x bar transpose, y bar, plus mu h, into sigma square, divided by sigma h square, into norm x bar square, plus, sigma square.

Remember sigma h square is the prior variance of the parameter h. Sigma square is the prior variance of the noise the zero mean (Refer Time: 16:04) noise v. Now what we are going to do is divide numerator and denominator by sigma h square, sigma, sigma square. So, what we are going to do, divide, divide both numerator and denominator. Divide denominator and numerator by sigma square, by the product sigma square, sigma h square, and therefore, what we will have is, h hat, that is the MMSE estimate h hat, this is equal to, you will get what you will get is basically, x bar transpose, y bar, divided by sigma, sigma square plus, mu h divided by sigma h square, divided by 1, divided by sigma h square.

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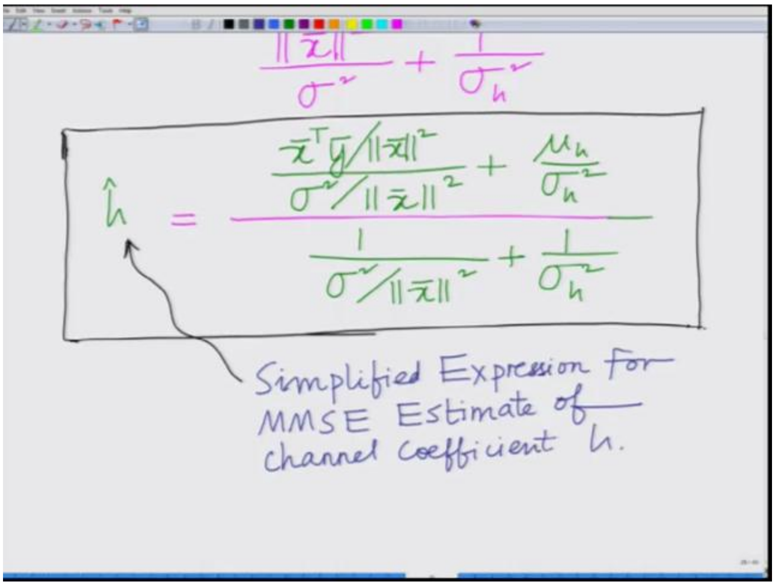

So, basically you will get, 1 divided by, norm x bar divided by sigma square, that is the first term. Norm x bar square, divided by, plus 1, divided by sigma h square, and I can simplify this further, just by the last step I can write this as, your h hat equals, x bar transpose, y bar, divided by norm x bar square, divided by sigma square, by norm x bar square, plus, mu h by sigma h square, divided by 1, by sigma square, by norm x bar square, plus 1 divided by sigma h square.

What we are saying is this is the final expression, simplified expression. And I am going to explain, what is the intuitive meaning behind this. Similar to what we have to seen in the case of the, wireless sensor network. What we have is a very beautiful interpretation for this expression. Now if look at this, the various quantities, so let us look at this is. What is this is? The simplified expression for the MMSE estimate of the channel coefficient h, this is the simplified expression, for MMSE estimate of the channel coefficient h. Simplified expression for the MMSE estimate of the channel coefficient h.

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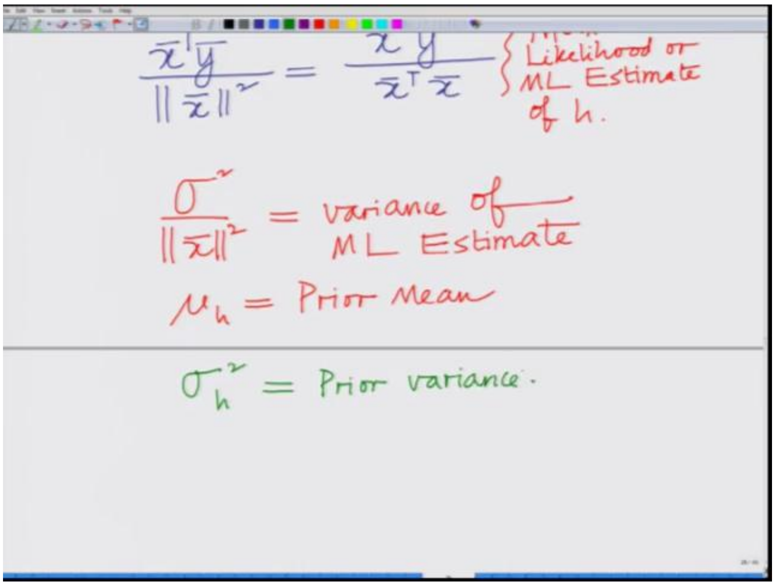

And now if you look at this look, at the all the quantity that have involved, we have x bar transpose, y bar, divided by norm x bar square, which is basically the same as x bar transpose, y bar, divided by x bar transpose, x bar, and you can see that this is the Maximum Likelihood estimate of the channel coefficient h. Some of you might not be familiar with this concept of Maximum Likelihood estimation, if you have not done the previous course, but anyway, does not matter. This is the same estimate, that is derived from the, based on the principle of Maximum Likelihood estimation of the channel coefficient h. So, this is the ml, or Maximum Likelihood, or ml estimate Maximum Likelihood or ml estimate of the channel coefficient h.

And what is this other quantity? Remember there is other quantity, sigma square, divided by norm x bar square. So, this quantity, we are saying, the ml estimate, this is the MLE, not this quantity, this quantity is the MLE, Maximum Likelihood Estimate. This quantify sigma square, divided by norm x bar square, this is the variance of the MLE. This quantity is the, this quantity equals, variance of Maximum Likelihood. This quantity is a variance of the Maximum Likelihood Estimate. And now, what is mu h? mu h is prior mean, and sigma h square is, prior variance, and this is also what know, this is the mean, remember we said h, the parameter, unknown parameter h, is random Gaussian in natures, its random in nature, it is distribute as, it has a Gaussian distribution prior Gaussian, prior density is Gaussian, it has mean mu h, and variance sigma h square, so mu h, equal to the prior mean, and sigma h square, equal, equal to the prior variance.

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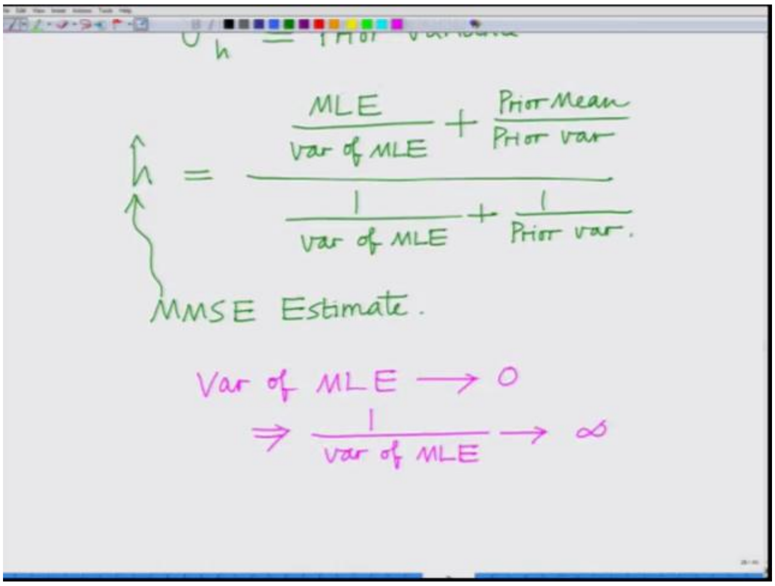

Therefore, we also have a another beautiful interpretation of the maximum law of the MMSE estimate, that is the ML Estimate, Maximum Likelihood estimate, by the variance of the Maximum Likelihood estimate, plus prior mean, divided by prior variance, divided by 1 by variance MLE, plus 1 by prior variance. So, what is this? This is that the mse estimate, this is the. So, what does this say, the MMSE estimate, is basically a linear combination of the Maximum Likelihood estimate, and the prior mean, that is mu h, in the inverse ratio of their variance. So, this is the beautiful interpretation.

And you can see there is several interesting implication of this. For instance, let us say, the variance of the MLE, variance of Maximum Likelihood estimate, is very small, it means it tends 0, implies, you can look at this, the variance of MLE is very small, then 1 by variance of MLE, 1 by variance of MLE, this quantity tends to infinity, therefore, you can see, this MLE by variance of MLE, this, this is the dominant quantity in the numerator, and 1 by variance MLE, is the dominant quantity in the denominator.

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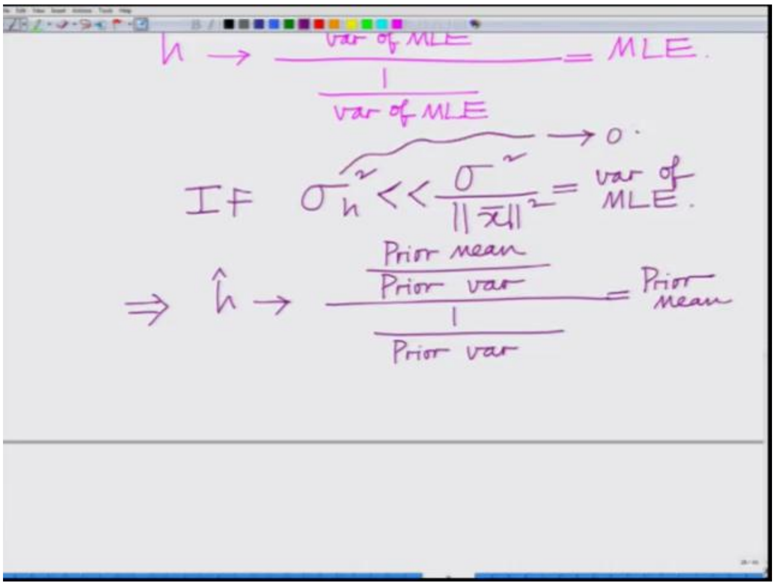

So, therefore, h hat tends to this quantity, MLE divided by variance of MLE, divided by 1 by variance of MLE. So, in fact, h hat tends to, in this scenario, h hat tends to simply the MLE, divided by the variance of MLE, divided by 1 by variance of the ML Estimate which is equal to the ML Estimate. And that is the very beautiful interpretation what we have, is when the variance of the Maximum Likelihood is estimate is very small, that is tending to 0, or in other words, it is very small in comparison to the variance of the prior, that is basically what we are saying, is sigma square, divided by norm x bar square, is very small in comparison to sigma square. This is the condition. Variance of MLE tends to 0, implies basically your sigma h sigma square, divided by norm x by square, is much smaller than the variance of the prior.

On the other hand, if on the other hand, you can see, if sigma h square, the variance of the prior, is much smaller than the variance sigma square, divided by norm x bar square, then what happens? In this case, sigma square divided by norm x bar square, is the variance of the Maximum Likelihood Estimate. Now in this case, you can see something equally interesting, in this case the opposite happens, in this case, 1 over prior mean, over prior variance for 1 over prior variance is very small. So, prior mean, over prior variance is very large, in comparison to MLE, divided by the variance of MLE. So, this quantity became dominant, in the denominator 1 over the prior variance, became dominant.

So, therefore, what we have, this implies, h hat tends to. In fact, prior mean divided by, prior variance, divided by 1 over, equals prior. In fact, it tends to prior variance, if sigma x square, if this quantity basically, tends to 0. Or the prior variance is very small. So, we have two interesting scenario. What means, what, what we have is, the prior variance is very small, if prior variance is very small, then sigma h square is very small, which means that prior mean, which means that the parameter h has the very small variance, around the prior mean, that is mu h. So, to high degree of accuracy, the prior, the parameter h, is very close to the prior mean mu h. That is what sigma x square close means therefore; the optimal MMSE estimate is in fact, the prior mean itself.

On the other hand, when you have sigma h square, or, or when you have sigma square dived by norm x bar square, is very close to 0, that is the variance of the Maximum Likelihood Estimate is very small, that mean, that basically, with the very high the degree of probability, or with the very high degree of accuracy, the parameter h, lies very close to its ML Estimate, that is x bar transpose, y bar, divided by norm x bar square. So, in this case, it converges to the Maximum Likelihood Estimate, x bar transpose, y bar divided by norm x bar square. So, that is the implication that is the practical implication and the inside, the meaningful inside that you obtain from this.

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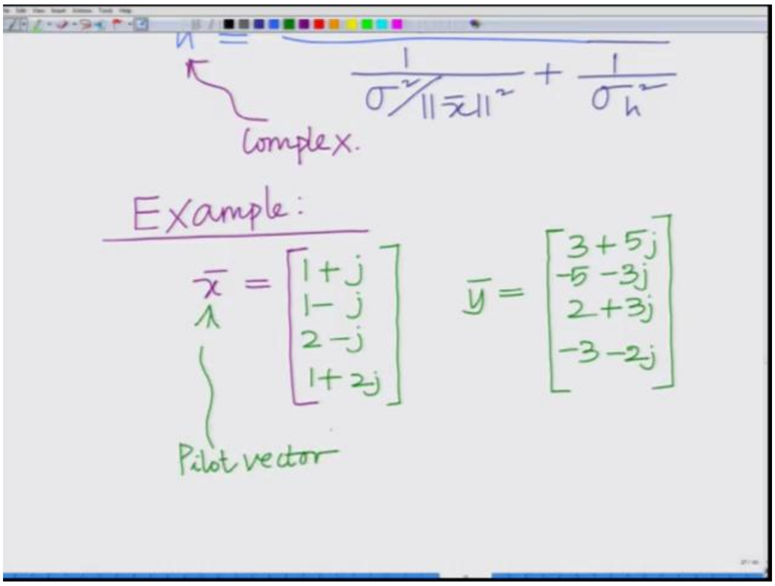

So, what you have is basically MMSE, is the weighted linear combination, of the ML Estimate, and the prior mean, and the weights are basically inverse, the weights are given by the inverse of the variance, which means the variance is large, the weight is small. So, naturally that is very intuitive, because if the variance is large, that means, there is a lot of inaccuracy, hence the corresponding weight has to be very small. And that is the real reason behind combining these two, in the inverse ratio, of their variances alright. So, that is the important thing.

And now, another small point, what do we do, when the quantity x bar, y bar, and h are complex quantities? When x bar comma y bar are complex, what do we do n this scenario, then the MMSE estimate, h hat is also going to be complex, h hat is going to be, all we have to do is replace by the transpose by the Hermitian. So, we going to have the MMSE estimate, is x bar Hermitian, y bar, divided by norm x bar square. So, nothing changes, except the transpose is replaced by the Hermitian, plus mu h divided by sigma h square, and of course, the denominator remains the same. 1 by sigma square, divided by norm x bar square, 1 by sigma h square. So, that is the expression for the MMSE, at this estimate is going to be complex. This estimate is going to be, this estimate is going to be a complex quantity, h hat going to be a complex quantity.

So, that is what happens, when the, for the MMSE. There is a leaner weighted combination of MML, ml and MMSE estimate, in the inverse ratio of variance. So, let me also write that down. Linear combination of, so this is the linear combination of ML, and MMSE, combination of MLE, comma prior mean, in inverse ratio, of in inverse ratio of, of their variances. So, this is an important principle to keep in mind.

So, linear combination of the ML Estimate Maximum Likelihood Estimate and the prior mean in the inverse ratio, of their variances. So, that is the important part, alright? Let us do a brief examples, let us do a small example, to understand this better. Let just do a small example. So, let me do a small example, for this, channel estimation, considering complex symbols. Consider x bar, pilot vector, equals, 1 plus j, 1 minus j, 2 minus j, 1 plus 2 j, y bar, equals, what is a our y bar? y bar is equal to, y bar is equal to, 3 plus 5 j, three plus 5 j, minus 5 minus 3 j, 2 plus 3 j, minus 3 minus 2 j, what is x bar? x bar is your pilot vector.

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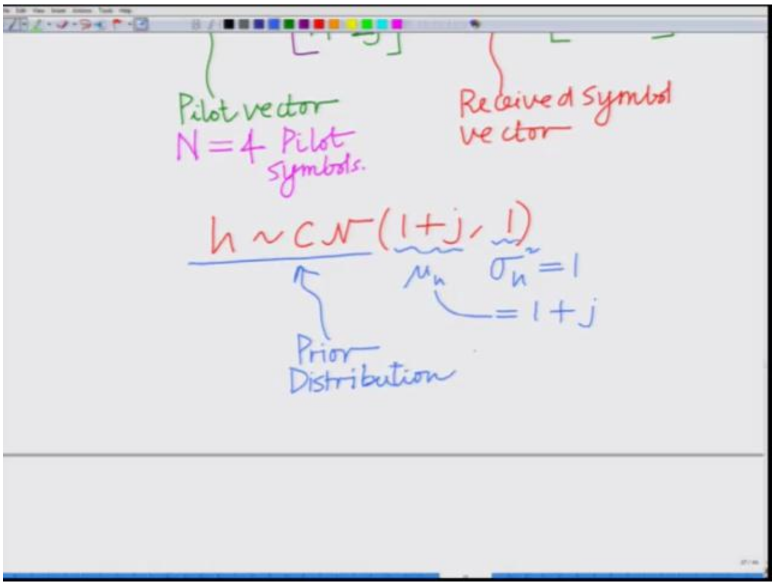

And you can clearly see, there are, pilot vector is four cross 1, implies there are N is equal to 4 pilot symbols, there are N is equal to 4 pilot symbols. Now y bar, this corresponds to the received symbols. This is vector of received symbols. This corresponds to the vector of received symbols.

Therefore, we have, the channel coefficient h, is proportional to the complex normal, let say channel coefficient h is distributed as is the prior density, is complex normal which mean, at 1 plus j, variance 1. So, what we have is this is 1 plus j, this is mu h, sigma h square, equals, 1, mu h equals, 1 plus j. This is the, what is this? This is the prior distribution. Since we are considering the complex scenario, the channel coefficient x is complex, it prior mean is complex, that is 1 plus j, and the variance sigma h square, is equal to 1. We are considering a complex channel coefficient estimation scenario.

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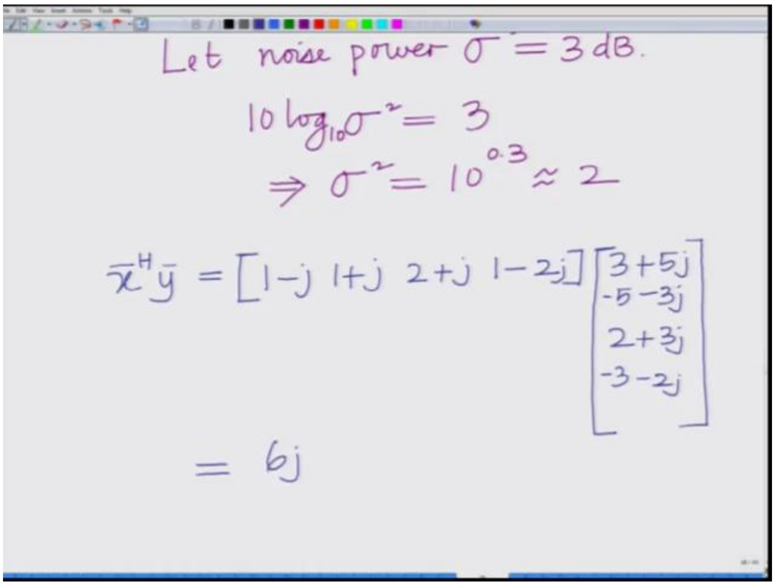

And we have already seen what is expression, the expression is so let the noise power, we need some other parameter. Let the noise power, sigma square be equal to, 3db, that implies, 10 log 10 sigma square, equals, 3 implies sigma square, equal to this, implies sigma square, equal to 10 power point 3, approximately equal to 2, The noise power sigma square, approximately equal to 2.

Next we want to find, what is this quantity x bar, Hermitian y bar, x bar Hermitian y bar. Remember for Hermitian of a matrix, you have taken the transpose of a matrix, and the complex conjugate of every element. So, column vector x bar becomes row vector, and you take the complex conjugate of every element. So, that becomes, 1 minus j, 1 plus j, 2 plus j, 1 minus 2j, into 3 plus 5j, minus 5, minus 3 j, 2 plus 3j, minus 3 minus, minus 3 minus 2j, and what is this going to be? This is going to be, you can calculate this, and this is going to be 6j.

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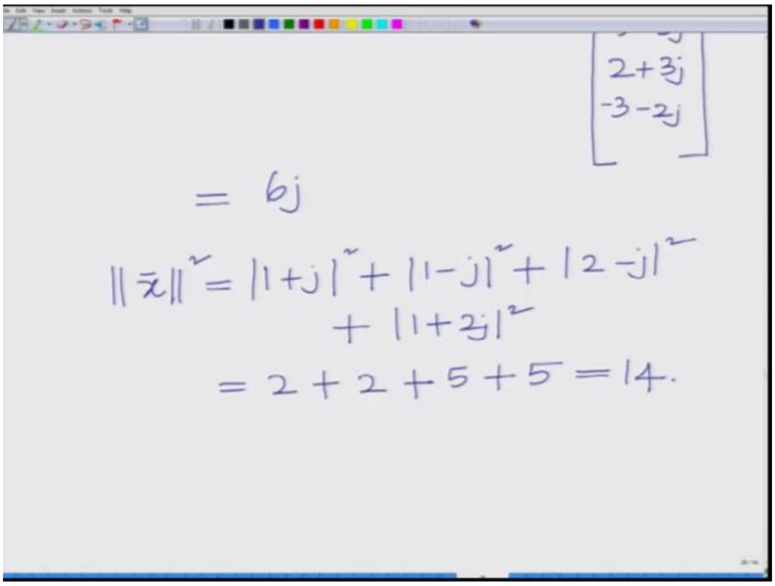

And the other quantity vd is norm x bar, square norm x bar square equals, magnitude 1 plus j square, this is magnitude 1 plus j square, plus magnitude 1 minus j square, plus magnitude 2 minus j whole square, plus magnitude 1 plus 2j whole square, which is equal to 2 plus, 2 plus, 5 plus, 5 plus, which is equal to 14.

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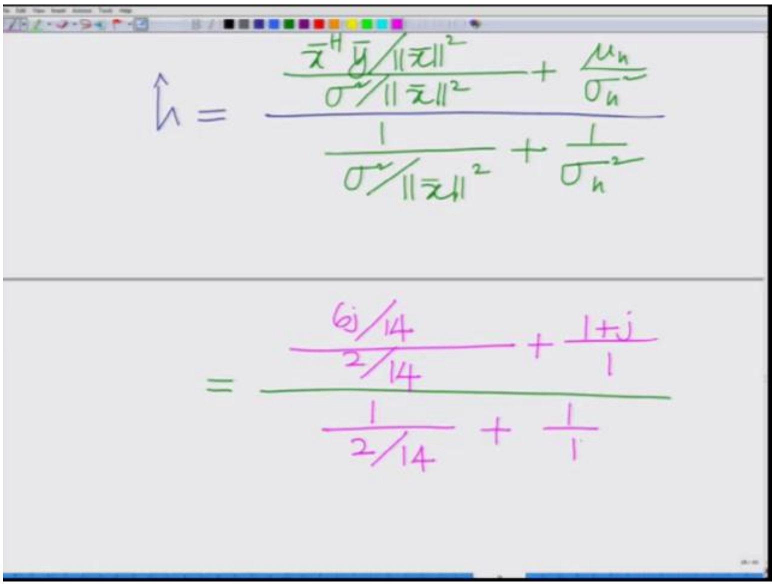

And therefore, our ML, or MMSE estimate, h hat, equals, lets write this expression again, once, x bar Hermitian, y bar, divided by norm x bar square, divided by sigma square, by norm x bar square, plus the prior mean, mu h, divided by sigma square, divided by 1 by, sigma square, divided by norm x bar square, plus 1 divided by sigma h square, which is equal to, now substitute the varies quantity that we have calculated above.

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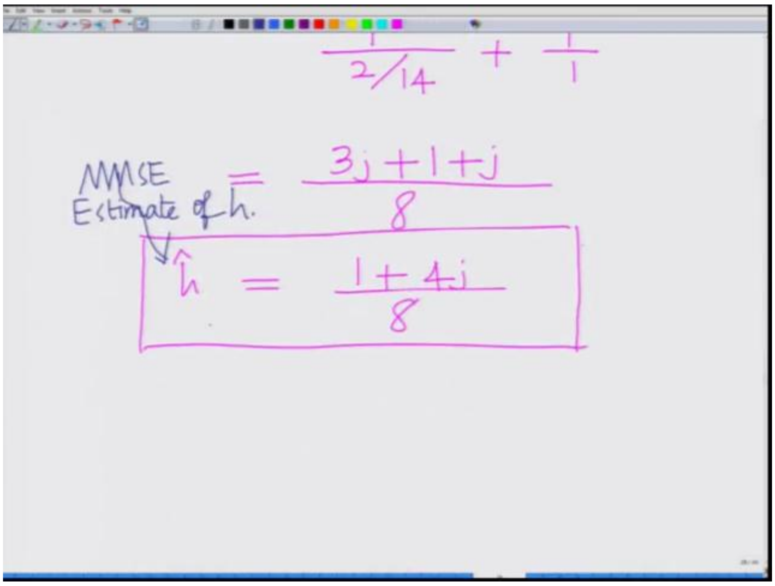

This is equal to 6j divided by 14, divided by well, 2 divided by 14, sigma square is 2. Norm x bar square is 14, plus mu h, is remember 1 plus j, divided by sigma h square, which is 1. 1 by, sigma square divided by norm x bar square that is 14, plus 1 divided by 1, that is sigma h square, and this is equal to, 3j plus 1, plus j, divided by 8, equals 1 plus 4 j, divided by 8.

What is this? This is the MMSE estimate of the channel coefficient h. This is the MMSE estimate of the channel coefficient h. So, you have considered a simple example, and derive the MMSE estimates, the channel coefficient h. So, we have done and just to summarize, in this module, what you have done is, we have simplified the expression of the MMSE estimate of the channel coefficient h, for a wireless communication scenario, derived previously. We have given an in intuitive interpretation, a nice intuitive interpretation, for this expression of the MMSE estimate of the channel coefficient h, and finally, we have demonstrated how to compute this MMSE estimate, using through a simple example.

So, we will close this module here, and we look at other aspect in subsequent modules.

Thank you very much.