Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications

Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 07
Simplification and Example of Minimum Mean Squared Error (MMSE Estimate for Wireless Sensor Network

Hello. Welcome to another module, in this massive open online course on estimation, MMSE estimation, for wireless communication systems. And what we are looking at currently is the MMSE estimate, of the parameter, in a wireless sensor network.

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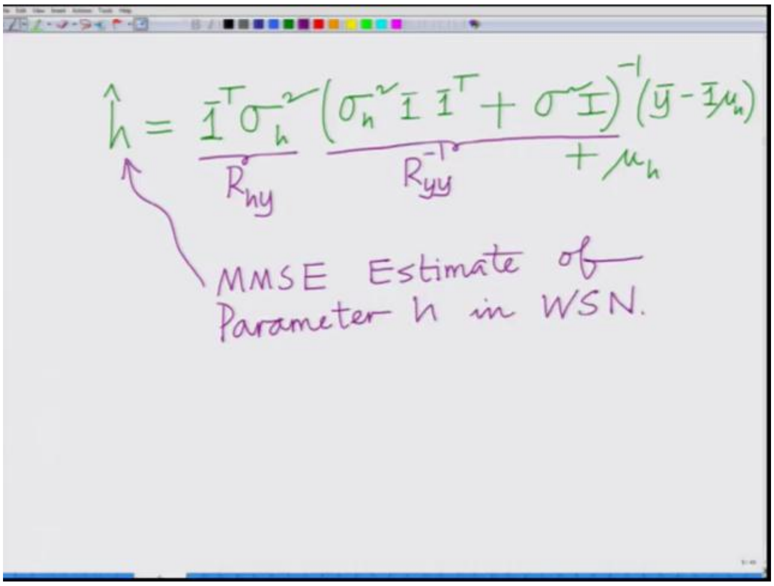

And we have derived the expression for the MMSE estimate as, h hat, that is the MMSE estimate of the parameter h hat, at the fusion center is, well that is, 1 bar transpose sigma h square times, sigma h square, into 1 bar 1 bar transpose, plus, sigma square I inverse, into y bar, minus 1 bar, times mu h, plus, mu of h, and this is basically, remember we said this is basically the expression, for Rhy and this is basically Ryy inverse, where Ryy is sigma h square 1 bar 1 bar transpose, plus sigma square times the identity matrix. What is this? This is the MMSE estimate, of the, MMSE estimate of the parameter h, of parameter h, in the WSN or in the wireless sensor network.

Now, what we are going to do in today's module, is further simplifies expression for the estimate of the parameter h. So, this is the complicated expression. So, we are going further simplify it. So, that it is easier to interpret, and easier to draw inference system. So, so what we are going to do is basically simplify the expression, and to simplify this expression, note the following thing. First let us start with this quantity over here.

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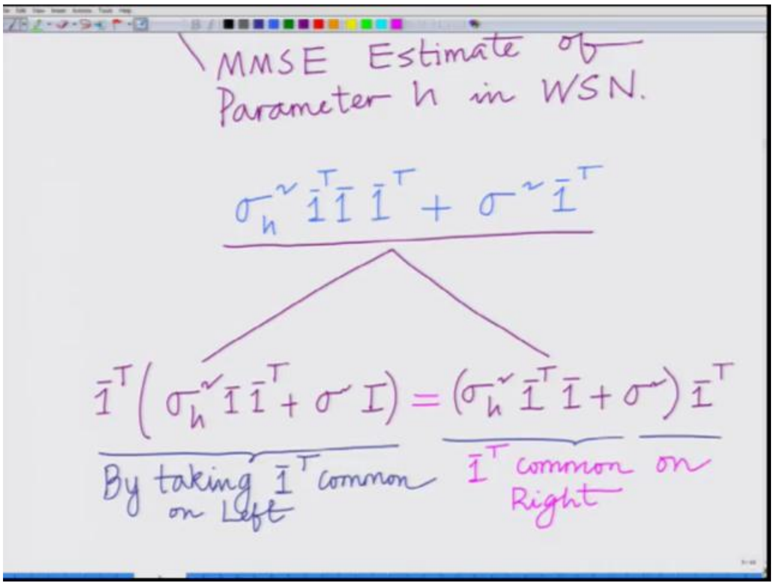

Let us start with this quantity, that is sigma h square, 1 bar, 1 bar, this is 1 bar transpose, 1 bar, 1 bar transpose, plus sigma square into 1 bar transpose. Now I can take, look at this quantity, I can simplify this into 2 forms, right? First, what I can do is, I can take 1 bar transpose, out on the left, and then this becomes, 1 bar transpose, sigma h square, in to a, or 1 bar transpose, times, sigma h square, 1 bar, 1 bar transpose, plus sigma square, times, identity. This is the expression I get by taking 1 bar transpose common on the left.

Now, what I am going to do is, I am going to take 1 bar transpose, outside on the right, and, then, what I get is sigma h square, 1 bar transpose, 1 bar, plus sigma square, times, 1 bar transpose. This is, this is, by taking 1 bar. Let me just write it down, by taking 1 bar transpose, common on the left side, by taking 1 bar transpose common on the right side. So, the same expression we have defined, we have simplified it in 2 different ways, that is, by taking 1 bar transpose common on the left hand side, and 1 bar transpose common on the right hand side.

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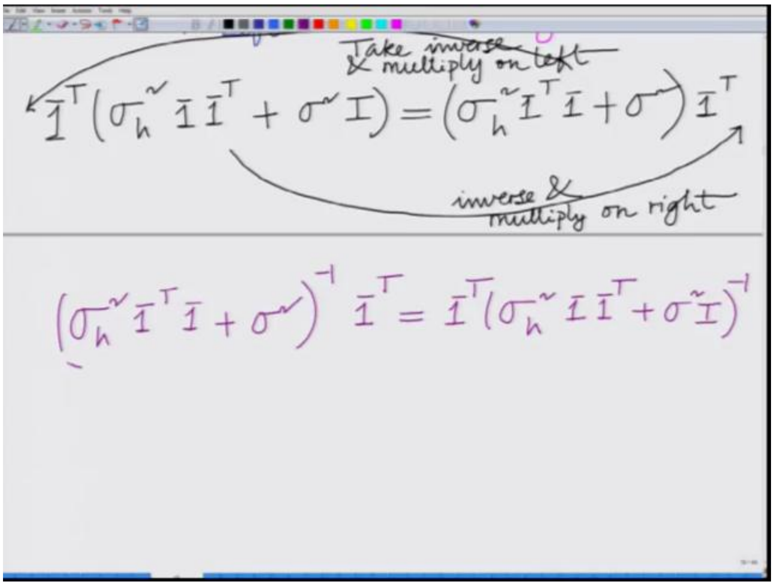

And therefore, these 2 quantities are basically one and the same. And therefore, these 2 quantities are equal. So, what we have, is let me write it down, 1 bar transpose, sigma h square, 1 bar, 1 bar transpose, plus sigma square times identity, is essentially equal to, sigma square, 1 bar transpose, into 1 bar, plus sigma square, into 1 bar transpose. And therefore, now what I can get is, I can take the inverse, and multiply this on the right, here. Inverse, and multiply on the right, of the expression on your left, and this term, I can bring it over here, take the inverse, and multiply it on the left, that is take inverse, take the inverse and multiply on the left. And when I do that, I can now simplify this interestingly as, sigma h square 1 bar, right, what do I have? I have sigma h square, 1 bar transpose, 1 bar, plus, sigma square, inverse, into 1 bar transpose, this is equal to, well, this is equal to, 1 bar transpose, into sigma h square, sigma h square, 1 bar, 1 bar transpose, plus, sigma square identity, inverse.

Now, what I am to going to do, is going to multiply both sides by the scalar quantities, sigma h square. I am going to multiply LHS and RHS by the scalar quantity, ah, sigma h square, which is the prior variance, of the parameter h, and of course, the equality remain unchanged. Because this is, the equality holds, because you are multiplying LHS and RHS by sigma h square.

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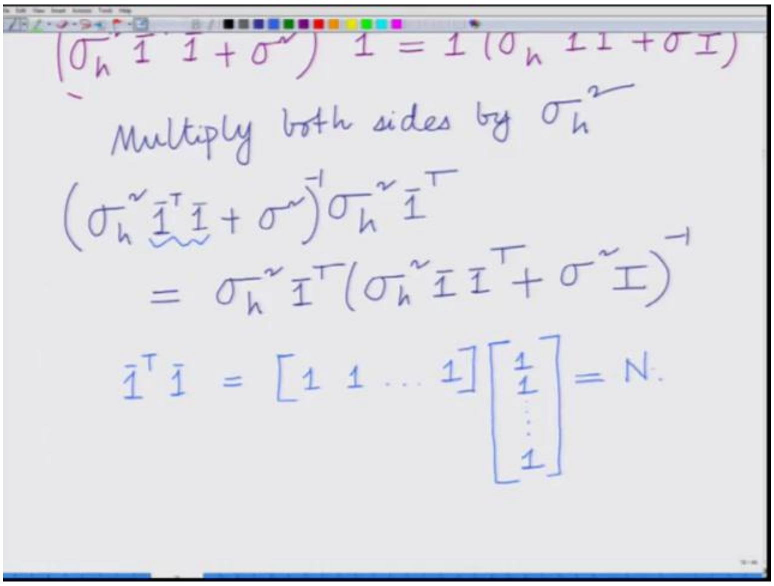

So, multiply both sides, and it is the scalar quantity, so I can place it appropriately, anywhere. So, sigma h square, 1 bar transpose, 1 bar, plus, sigma square times, 1 bar transpose, equals, and I am multiplying by sigma h square. So, I can just insert the sigma h square over here, sigma h square into 1 bar transpose, this is equal to the RHS which is, sigma h square, 1 bar transpose, sigma h square, 1 bar, 1 bar transpose, plus, sigma square identity, inverse.

Now, observe this quantity 1 bar, 1 bar, 1 bar transpose 1. Now observe this quantity, 1 bar transpose, 1 bar is simply, what is this? It is the row vector, the end dimensional row vector of all 1s, times, the end dimensional column vector of all 1s, and which is simply equal to N. 1 bar transpose, 1 bar, is equal to N, and therefore, what we have now is interestingly this quantity here.

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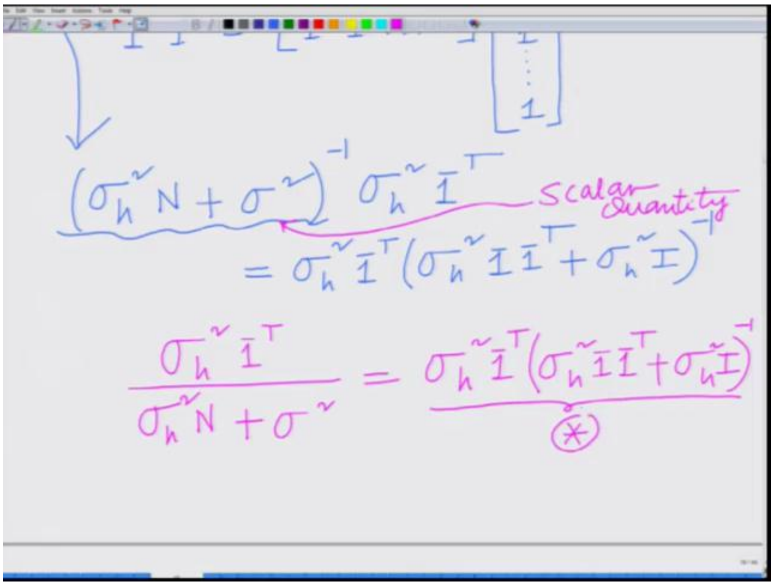

This is basically sigma h square, into N plus it is a scalar quantity, you can observe that. Sigma h square N, plus sigma square into 1, eh, sigma square inverse, into sigma h square 1 bar transpose, equals, sigma h square, 1 bar transpose, sigma h square, 1 bar transpose, plus sigma h square, identity, inverse. Now this is the scalar quantity, look at this, this is simply a scalar quantity. So, inverse is the reciprocal. So, this is the scalar quantity. And therefore, we have sigma h square, divided by, sigma h square N, plus sigma square, which is equal to, sigma h square, 1 bar transpose, sigma h square, 1 bar, 1 bar transpose, plus sigma h square, identity, inverse.

Then now if you observe, the quantity on the right hand side, let say this quantity, lets denote this by star, this quantity is the same as, if you look at this, this quantity is the same as this quantity, this quantity, observe that. This is the star quantity, which is simplified below, right? And therefore, now if you look at this, now I can replace this, and now, I can simplify the expression.

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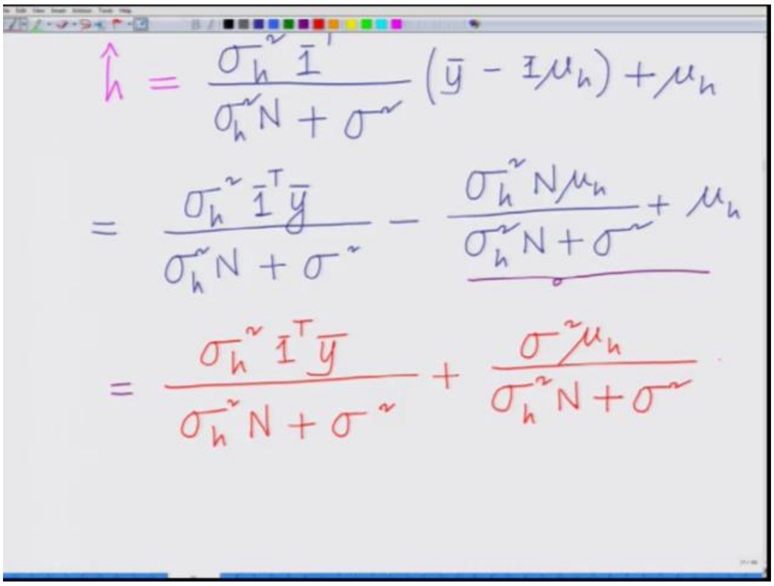

So, if in the earlier expression for h hat, I can replace the star quantity, by our simplified expression, here. And now, MMSE estimate for h hat can therefore be derived as, sigma h square, 1 bar transpose, divided by sigma h square N, plus, sigma square, sigma square, times, y bar, minus 1 bar mu h, plus mu h. So, this is after substitution, and therefore, now look at this, I have sigma x square, 1 bar transpose, into y bar. So, that becomes sigma h square, 1 bar transpose, into y bar, divided by sigma h square N, plus sigma square, minus, sigma h square 1 bar transpose 1 bar, look at this, sigma h square 1 bar transpose 1 bar, that is N. So, this is sigma h square, 1 bar transpose, 1 bar, which is N times, mu h, divided by sigma square N, plus sigma square. In fact, this should be, this should be, sigma h square in the denominator.

So, this is sigma h square, sigma h square, plus, of course, mu h, and now if you look at this, the term on the right, look at this I have sigma h square N mu h, minus sigma h square N mu h, plus, mu h N sigma square. So, what I will get if I simplify this, is basically, sigma h square, 1 bar transpose y bar, divided by sigma h square N, plus, sigma square, plus, sigma square, mu h, divided by sigma h square N, plus sigma square, which is equal to therefore, this is equal to sigma h square, 1 bar transpose, y bar, plus sigma square, mu h, divided by sigma h square N, plus sigma square, this is the simplified expression.

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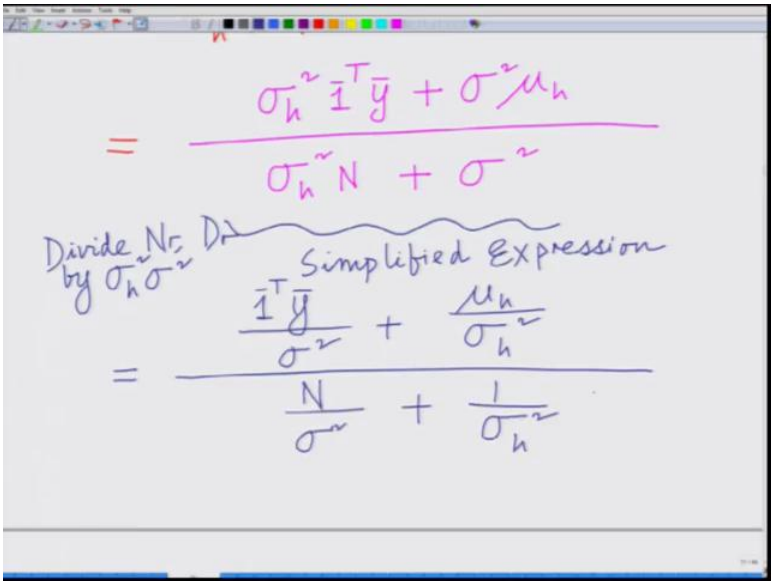

So, this is the simplified expression. It is a much simpler form, because it does not involve any matrix inverses, like we have seen, simplified expression, and we are going to simplify it further, but this is a sort of an intermediate simplified expression, of the MMSE estimate, all right? And now, I am just going to put it in the form, in a slightly better, improved, slightly more intuitive form. What I am going to do is, I am going to divide, numerator and denominator, by sigma h square sigma square. So, divide, numerator comma denominator, by sigma h square, by the product sigma h square sigma square. What that will give me is, 1 bar transpose, y bar, divided by, 1 bar transpose y bar, divided by, sigma square, plus mu h, divided by sigma h square, divided by 1, or N, divided by sigma square, plus, 1 divided by sigma h square.

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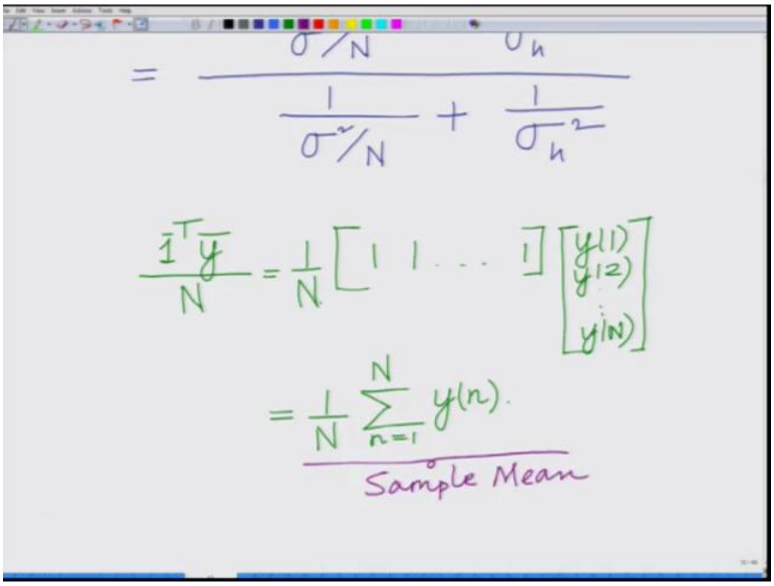

I am going to now write this in a slightly more convenient form, and going to, 1 bar transpose y bar, divided by N, divided by sigma square, divided by N, plus, mu h divided by sigma h square, divided by 1, divided by sigma square, divided by N, just slightly different way of writing it by 1, divided by sigma h square.

Now look at this, this is an interesting, this has a very interesting interpretation, let us look at this, term by term, what does this quantity 1 bar transpose y bar divided by? Let us dissect this expression, what is this quantity 1 bar transpose y bar divided by N? Now you will realize that, is something very interesting 1 bar transpose, y bar, divided by N, divided by N. This is equal to 1, 1 row vector of all 1s, times, y 1, y 2 up to y N, 1 divided by N, which is nothing but, if you look at this, this is the very interesting interpretation, this is summation, 1 over N, N equal to 1 to N, y of n, and then if you look at this, this is nothing, but the average of the samples, this is the sample mean, and this is also, basically, the Maximum Likelihood Estimate. This is also, if you remember, some people who have done the previous course might remember, this is the, this is the maximum likelihood.

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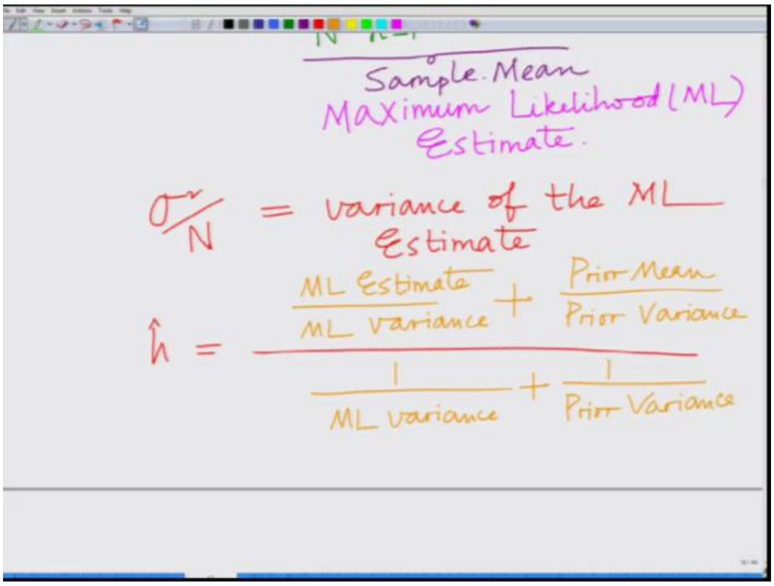

So, the first component 1 bar transpose y bar, divided by N, it is nothing, but 1 over N, summation, N equal to 1 to capital N, y of N is basically, the average of the observations, this is known as the sample mean. This is also the Maximum Likelihood Estimate. And now, let us look at this, sigma squared divided by the N in the numerator, this is nothing but the variance of the Maximum Likelihood Estimate, sigma square divided by N, this is equal to the, this is the variance of the ML estimate.

Now, you look at this, mu h, mu h is the mean, of the prior, and this is the variance of that, is the mean, mu h is the mean of the prior, remember we said the parameter h is Gaussian, with mean mu h and variance sigma h square. So, the mean of the prior is mu h, variance of the prior is sigma h square. Therefore, you have a very interesting interpretation for this. So, h hat equals, what does that equal to? It equals the ML estimate, divided by ML, variance plus, prior mean, divided by prior variance, divided by 1 by ML, variance, plus 1 by prior variance, so it has a very beautiful interpretation. What did is doing is, basically we have the ML estimate, Maximum Likelihood Estimate, and we have the, the prior mean, that is mu h, that can be treated as another estimate, on the parameter h, and then you combining, or you are performing a weighted combination of these 2 estimates, the ML estimate, and the prior mean, in the inverse ratio of the variances. That is your waiting each, that is your weighing, each estimate, by the inverse of the variance, because remember, if the variance is high, the estimate is less reliable.

Therefore what you are doing is you are doing weighting by the inverse of the variance. So, if the variance is high, the inverse of the variance is low, therefore, the corresponding weight is less. So, what we are doing is, you are taking these 2 estimates, and combining them in the inverse ratio, of the variances, that is the beautiful interpretation in the MMSE estimate has. So, and that is the expression. So, therefore, the final expression for the MMSE estimate, h hat, let us write it down again, the final expression, this is the very interesting expression.

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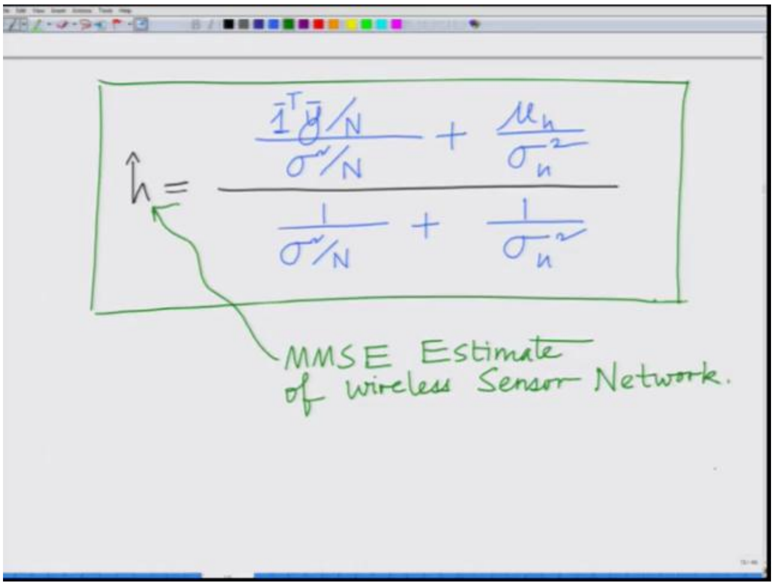

Although it looks, initially, it looks very complex, it as the very beautiful and a very interesting interpretation. 1 bar transpose y bar divided by N, divided by the ML variance, plus, the prior mean, divided by the prior variance, divided by inverse of the ML s variance, plus, inverse of the prior variance.

So, it has a very, very beautiful interpretation, and one has to appreciate, this beautiful interpretation, of the MMSE estimate, in the wireless sensor network. So, this is the MMSE, MMSE estimate for the wireless sensor network, where the parameter h, is being sensed by the wireless sensor network. So, this is the simplified expression for the MMSE estimate.

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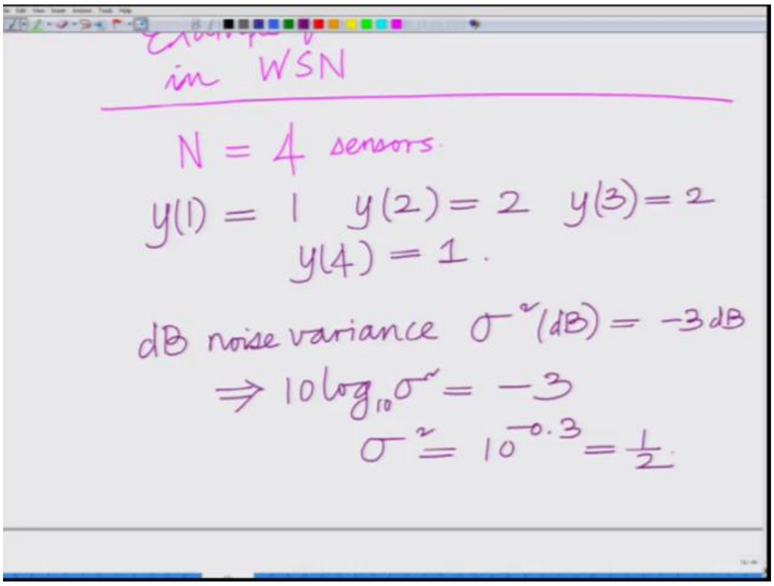

Let us do a simple example, to understand this. So, simple example for wireless sensor network let us do a simple example to understand this. Let us consider N equal to 4 measurements or N equal to 4 sensors, which naturally means, that there are N equal to 4 measurements. Let the measurements be denoted by, let that measurement be denoted by, y 1 equals 1, y 2 equals 2, y 3 equal to 2, and y 4 equals 1. Then the ML estimate, and also let say, the noise variance, d b noise variance, that is sigma square, let say the d b noise variance, sigma square, in d b equals, well this is equal to minus 3, d b, which implies that basically your, 10 log 10 sigma square, equals minus 3, which implies sigma square, equals 10 to the power of minus point 3, minus point 3, equals half.

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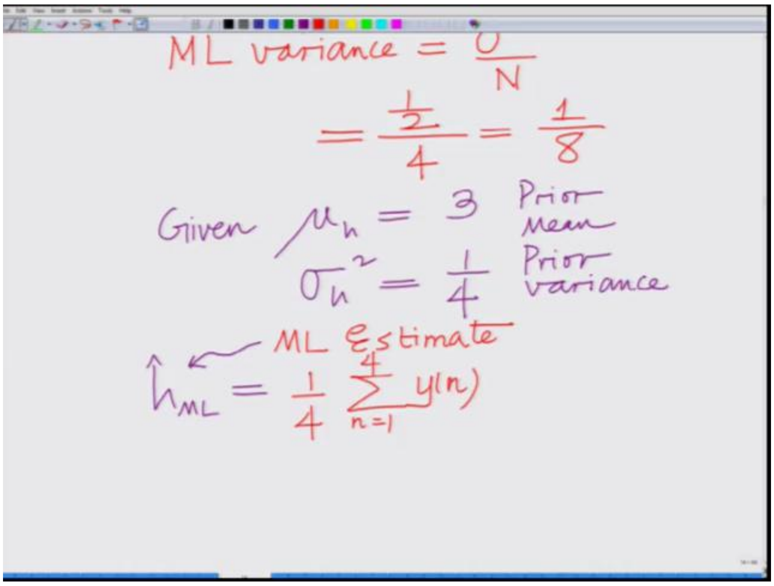

And therefore, the ML variance, remember the ML variance, equals, sigma square, divided by N, that is half, divided by N, equal to 4, that is 1, divided by 8, that is a ML variance. Sigma square 1, divided by N, half divided, eh, sigma square divided by N, that is half divided by 4, that is 1 divided by 8, that is the ML variance.

Now, let us look at the prior, let say, the prior given prior mean, let say prior mean, given of course, this information has to be given, mu h equal to 3, what is this? This is your prior mean, and sigma h square, equals, 1 over 4, this is the prior variance, and. Now, therefore, the, and now, let us compute the ML estimate. Remember that is what we need h hat ML, what is this? This is basically your ML estimate, that is the simply the sample mean, that is equal to, 1 over 4, summation, N equal to 1 to capital N, which is 4, of y N, the observations y N, which is basically, 1 over 4, times 1, plus 2, plus 2, plus 1, which is equal to, well, that is 6 divided by 4, which is equal to 3, divided by 2.

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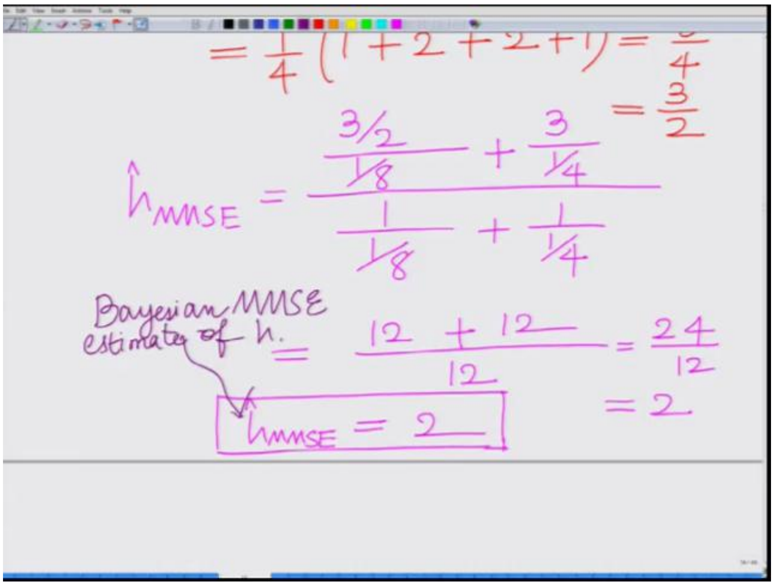

Therefore, the MMSE estimate, we have already seen the ML variance, therefore, MMSE estimate h hat, MMSE, let me write it a little bit more clearly, h hat, MMSE equals, now we are all the required quantities, that is the ML estimate, divided by the ML variance, that is 1 over 8, plus the prior mean, 3, divided by the prior variance, that is sigma h square which is 1 over 4, plus 1 over the ML variance, plus 1 over the prior variance, which is basically equal to 12, plus 12, divided by 12, which is 24, divided by 12, which is equal to 2. Therefore, in this simple example, the MMSE estimate of the parameter h hat MMSE, that is equal to 2, where this is the MMSE.

Let me just write that again this is the Bayesian, this is the Bayesian, Bayesian MMSE, Bayesian MMSE estimate, of the parameter h. This is the Bayesian MMSE estimate in the parameter h. So, we have done, in this module, is we have simplified the expression of the MMSE estimate, of the parameter h, in the wireless sensor network, and finally, we have also seen a simple example, of how to compute, this MMSE estimate for a simple scenario with N equal to 4 observations, and what is interesting, is we have seen that this MMSE estimate as the very beautiful interpretation. It is the combination of the ML estimate, the classical ML estimate, which is the sample mean, and the prior mean, in the inverse ratio; it is the linear combination, a weighted linear combination of them, in the inverse ratio, of their variances.

So, with this module, we stop here. And therefore for this module and we will continue with other aspects in subsequent modules.

Thank you very much.