Fiber Optics

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Lecture – 09
Propagation in Infinity Extended Dielectric – I

We have seen that ray theory has its limitations. It cannot accurately predict the propagation characteristics of an optical fiber, particularly when the light confinement dimensions are of the order of or comparable to the wavelength of light. In that case we will have to use wave theory. In wave theory light is treated as an electromagnetic wave. And therefore, we need to understand how these electromagnetic waves propagate in an optical fiber. Before doing that we would like to first understand how an electromagnetic wave propagates you know free space or in a dielectric medium which is of infinite extent for example, this room. So, first we would understand how light propagates in this room or in an infinitely extended dielectric medium.

So, we will do this in this lecture. Since we want to understand propagation of light in infinitely extended dielectric medium, and light is an electromagnetic wave. So, we should first consider the Maxwell's equations in infinite dielectric medium.

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So, the simplest case if we take, then it is a homogenous linear isotropic charge free and current free dielectric medium. One example of such a medium is glass. What is homogenous medium? Homogenous medium means that the refractive index of the medium is the same at every point. It does not depend upon *x*, *y* and *z*. Linear medium means that if I propagate light of frequency ω ~~omega~~ then it propagates as frequency ~~omega~~ itself, it does not generate new frequencies. Or if I propagate light in a medium then the refractive index of the medium remains independent of the intensity of light. So, this is a linear medium. Isotropic means that if I have ~~2~~ two independent orthogonal polarizations for example, this and this. Then these polarizations see the same refractive index of the medium. And since it is a dielectric medium, so there are no free charges and no ~~of~~ free currents.

So, let me write down the Maxwell's equations, which is the starting point for us. So, the first Maxwell's equation is $\vec{\nabla}.\vec{D}=0$ ~~del dot D is equal to 0~~, which is nothing but the differential form of Gauss's Law. The second equation is $\vec{\nabla}.\vec{B}=0$ ~~del dot B is equal to 0~~, which simply tells that magnetic monopoles do not exist. Third is $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ ~~del cross E is equal to minus del B over del t~~, which is nothing but Faraday's law in differential form. And the last one is $\vec{\nabla}\times\vec{H}=\frac{\partial\vec{D}}{\partial t}$ ~~del cross H is equal to del D over del t~~, which is nothing but Ampere's law. Apart from these 4 four Maxwell's equations we also have constitutive relations which are $\vec{D}=\varepsilon\vec{E}$ ~~D is equal to epsilon E~~. And $\vec{B}=\mu\vec{H}$ ~~B is equal to mu H~~ if I consider a dielectric medium, such as glass then it is a non-magnetic medium and in a non-magnetic medium $\mu=\mu_0$ ~~mu is approximately equal to mu naught~~.

So, at some places I will use this approximation also. I will use the convention throughout my lectures that curly letters wherever I use curly letters it means that they are the functions of *x*, *y*, *z* and *t*, while the straight letters do not have any time dependence they are the functions of only the special coordinates. So, what I want to do? I want to find out how electromagnetic wave propagates in a medium. And electromagnetic wave has associated electric field and magnetic field. So, what I want to do is basically I want to find out how the electric and magnetic fields associated with the light waves, vary with special coordinates and with time. For that I need to form a differential equation in E and a differential equation in H, which can tell me how E and H ~~very~~ vary with *x*, *y*, *z* and *t*.

In order to form the differential equation, let me again look at the 4 four Maxwell's equations and then let me do some mathematical manipulations. So, what I do? I take the curl of equation 3. So, I get del cross del cross E is equal to minus del del cross B over del t. $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t}$.

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Now I use the vector identities which gives me del of del this is equal to del cross del cross is equal to gradient of divergence of E minus del square E. And if I look at this the del cross B and I look at equation 4, if I multiply here by mu and here also then it becomes del cross b. So, del cross B is mu times del D over del t, So I put it here. Now since this is a homogenous medium and in a homogenous medium what I have del dot D is equal to 0 and D is equal to epsilon E. So, in a homogenous medium since epsilon is not a function of *x, y* and *z*, then that epsilon will come out of this del divergence operator. Which means that in a homogenous medium del dot D is equal to 0 will translate to del dot E is equal to 0.

So, this term goes to 0. And again I put D is equal to epsilon E then this equation simply becomes del square E is equal to mu epsilon del 2 E over del t square. So, here you can see that this del square is nothing but del square over del *x* square plus del over del *y* square plus del square over del *z* square. So, this simply tells me that how E varies with *x* *y* and *z* and with time. So, I need to solve this differential equation. And get the functional form of E. So, in order to solve this equation, first what I would do I will consider a very simple case, simple case of one dimension. Which means that I assume that E varies only with *z* and *t*, if I assume this then this equation will now become del 2 E over del *z* square is equal to mu epsilon del 2 E over del *t* square.

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And since E is the vector E*x* *x* cap plus E*y* *y* cap plus E*z* *z* cap, which means that this is note single equation, but it constitutes 3 equations - one in E*x*, one in E*y* and one in E*z*.

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And so, so I have equation 3 equations in scalar components. So, if I draw this vector sign and write the scalar equation then it would be del 2 E over del z square is equal to mu epsilon del 2 E over del t square where E is nothing but it can be either Ex or Ey or Ez. So now, I need to solve this equation. How do I solve it? Well since E is a function of z and t and I also see that mu epsilon is independent of z and t, then I can use the method of separation of variables. What is the method of separation of variables? Well, I can represent this E of z t is equal to E of z and t of t. So, I separate them out. If I put this E into this equation, then I get an equation t times del 2 E over del z square is equal to mu epsilon Ez del 2 t over del t square.

I divide this with this E times t and then I transform this equation into this form. What I have done essentially? I have separated out the variables. On the left hand side I have terms which contain only z, which are the function of z only and on the right hand side I have terms which contain t which are the function of t only.

Now, to solve this what I do? The most natural thing is that I quit each of them to some constant. And since it is a second order differential equation, then that constant I will take in the form of his of a square. So, that I avoid square roots.

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So, I quit it to some constant k square. Let us now first solve the t part with this please see that k, k is simply a constant, a purely a mathematical constant. What does it signify? Physically we will see later on. So, if I now take this t part, then it is mu epsilon del 2 t over del t square is equal to k square t, and let me write this mu epsilon as 1 over v square, mu epsilon is equal to 1 over v square. Why I am doing this? Because this will then come here and I am taking in the form of a square, so that I have everywhere square itself.

So, this will become now this and *v* square will come this side. So, what I do? I have now *v* square *k* square I represented by another constant omega square. So, in this way the *t* equation becomes del 2 *t* over del *t* square is equal to omega square times *t.* So, please again look that this *k*, this *v* and this omega there constants. I write now do not have any physical and interpretation for them, but later on I will see what is the physical interpretation. What is the solution of this equation now? I can immediately see that the solution of this equation is of the form *e* to the power plus minus omega t. Which means that s t goes to plus or minus infinity the solution blows up which means that this is not a physically viable solution. It is a solution it is a mathematically correct solution, but I do not have any use for such kind of solution because it is not physically viable. What kind of solution I am looking for? Well, I am looking for a solution which represents a wave. And a wave we will have an oscillatory solution, which means sin omega t or cosine omega t or a combination of these.

So, I can immediately see that if I get a solution which is of the form of *e* to the power plus minus *i* omega *t* then it is an oscillatory solution. And for that I need to have minus omega square here, which means that I should have minus *k* square here. So, the most natural choice that occurred to me to take this as plus *k* square, does not give me any physically a viable solution. So, I take this as minus *k* square. So, I take it as minus *k* square and now if I do the same thing and get the solution then it is of course, e to the power plus minus *i* omega *t*, which is an oscillatory solution.

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Now let us look at z part. So, z part is 1 over E del 2 E over del z square is equal to minus k square or del 2 E over del z square is equal to minus k square Ez. Then the solution of this equation is simply of the form e to the power plus minus i kz.

So, if I now combine these z part and t parts, then the complete solution is Ez t is equal to E 0 e to the power i plus minus kz plus minus omega t. This is the most general solution, where E 0 is the constant. It does not depend upon x, y, z or t. So, this is the most general solution of this equation. And this represents a wave. Now I can choose the sign shear appropriately So that it can represent a wave in a particular fashion. For example, if I choose the signs like this I represent it like this E 0 e to the power i omega t minus kz then this is the wave propagating in positive z direction. How it is propagating in positive z direction? I will see. What else I now see here is if I take a particular position z. If I fix z, let us say z is equal to 0. And I look at the solution in time then I see that the solution is oscillating in time. And it is oscillating in time with frequency omega after a certain time which is ~~which is~~ 2 pi over omega, it comes back to the same position.

So, omega is nothing but the frequency or angular frequency of oscillation. So now, I interpret this omega which I had represented here, as the frequency. Now let us take a snapshot of this, is snapshot of this means that I freeze frame in time. Let us I fix, let us take that time is t is equal to 0, then the solution is e to the power minus i kz. Now if I look at this I plot it. So, it would be an oscillatory wave like this. In z a sinusoidal function in *z*. And I see that it repeats itself after a distance 2 pi over *k*. Then *k* is nothing but the wave vector. So, here the *k* which I had taken just this constant purely mathematical constant now I see that it is nothing but the wave vector. And the omega which came out as *v* times *k* is nothing but the frequency. What is *v* I still need to see? Let us examine the nature of this wave what kind of wave it is. If I look at this phase of this wave then I find out what are the surfaces of constant phase, then what I find that if I put omega *t* minus *kz* is equal to constant so that I get the surface of constant phase.

Then at a particular time *t* I get the surface *z* is equal to constant. Z is equal to constant is an equation of a plane, which means that the surface of constant phase is a plane. And so, this kind of wave is known as plane wave. Let me find out at what velocity this surface of constant phase is moving. So, for that what I do I take the derivative of this, I differentiate this. When I differentiate this then I get omega D*t* minus *kz* is equal to 0, which means D*z* over D*t* is equal to omega over *k* and omega over *k* is nothing but *v*, which means that *v* which I had from here is nothing but the velocity of the wave, the phase velocity. So, this is a plane wave moving with phase velocity omega over and that omega over k is nothing but v; the constant which we represent it here.

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Now, let me generalized this for 3 D case. So, the 3 D wave equation is this. And so, the solution would be E *x y z* t is equal to E 0, E 0 is again a constant times e to the power *i* omega t. So, the t solution remains the same. And since now I have *xyz* all of them. So, the solution would you minus *kx x* minus *ky y* minus *kz z*. What is *k*? *k* is nothing but *kx* x cap plus *ky* y cap plus *kz* z cap. And if I represent the position vector in vector form then it is *x x* cap plus *y y* cap plus *z z* cap. So, this is nothing but this is nothing but *k* dot r. So, I represent it as E 0 *e* to the power *i* omega *t* minus *k* dot *r*. And from here you can immediately see that *k* square is *kx* square plus *ky* square plus *kz* square.

Now, for a given *k* I can have infinite sets of *kx ky kz*. So, infinite sets of *kx ky kz* can give me the same value of *k*. What is the meaning of those infinite sets? If you look at this is k dot r, in what direction this is moving? It is moving in one particular direction, which is represented by the values of *kx ky* and *kz*. *kx ky* and *kz* are nothing but the projections of vector *k* on *x y* and *z* axis. So, the values of *kx ky* and *kz* will give me at what angle this wave is moving. And with the propagation constant *k* the magnitude of propagation constant is always this. So, for a given magnitude of propagation constant *k*, I can have various angles possible. So, this tells me that if a light wave is allowed to go in an infinitely extended dielectric medium, then it can go in any direction. There is no restriction on it, it can go in this direction, this direction, this direction, this direction, this direction, this direction any direction is possible for the same value of *k*.

I can do the same analysis for the magnetic field. And I will get the solution as H *x y z t* is equal to H 0 *e* to the power *i* omega *t* minus *kx x* minus *ky y* minus *kz z* or H 0 *e* to the power *i* omega *t* minus *k* dot *r*. So, for a light wave I have got now the associated electric field which is represented by this, and associated magnetic field which is represented by this. These are of course, by scalar components. The same solutions are for E *x y z* and H*x* H*y* H*z*. Well, so let me now consider a wave which is going in certain direction and let me choose does that direction as *z* direction. If I choose the direction is *z* direction, then mathematically I can write the electric field associated with this as E 0 *e* to the power *i* omega *t* minus *kz*. Of course, this constitutes 3 equations, one is in E*x* another is in E*y* and yet another one is in E*z*.

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Similarly, the magnetic field can be written like this, where H is Hx x Hy y and Hz z cap. Now let me do one thing with this, let me take the divergence of this and since this is a homogenous medium, then del dot E would be equal to 0. If I put del dot E is equal to 0 it means that del Ex over del x plus del Ey over del y plus del Ez over del z is equal to 0. If I pick up Ex components from here then it would be E 0 x e to the power i omega t minus kz. E 0 x is the constant and here you do not see any term of x which means del Ex over del x is equal to 0. Similarly del Ey over del y is equal to 0, which means that if these 2 are 0 then this implies that del Ez over del z should be equal to 0, one thing. Second I do del dot B is equal to 0 I use this Maxwell's equation del dot B is equal to 0. So, similarly it will also give me the del Hz over del z is equal to 0.

So, these solutions of the differential equations the wave equations which I got earlier. Now if I put these solutions into these Maxwell's equations then they give me del Ez over del z is equal to 0, and del Hz over del z is equal to 0. What does it mean? If I look at Ez, from here then Ez would be E 0 z E 2 the power i omega t minus kz. If I look at Hz from here then it would be H 0 z e to the power i omega t minus kz. Now for this to be 0 there are 2 possibilities. One is Ez is constant and another is that the amplitude of Ez itself is 0. Let me first look at the first possibility that Ez is constant. If Ez is constant which means this whole thing is constant, E 0 z itself is a constant. So, which means that this term has to be a constant, if this term is a constant then there is no wave. Because then even the x component and y components they will become constant they will not vary with z and t.

So, there is no wave solution, which means that I cannot put e to the power i omega t minus kz is constant. So, in order to satisfy this equation the only possibilities E 0 z 0 which means that Ez is equal to 0 that is there is no longitudinal component because z is the direction of propagation itself. So, Ez is nothing but the longitudinal component. So, there is no longitudinal component. Similarly this equation gives me that H 0 H should be equal to 0, or in this way Hz is equal to 0. So, there is no longitudinal component of magnetic field also which means that the solutions of the wave equation that I got are the waves which do not have any longitudinal components, which means that they are the transverse waves.

Now if the wave is propagating in z direction, then there is no Ez there is no Hz. ~~It~~ Which means that E cannot vibrate along z and H cannot vibrate along z. So, they can only vibrate in x and y directions. That is they can vibrate in a plane perpendicular to the direction of propagation if z is the direction of propagation which is the direction of this pointer then and then this is the transverse plane.

So, E and H can only vibrate in this plane. So, which means that your if this is the plane, then E can vibrate along this or along this or along this or along this. And similarly H can vibrator along this along this along this, what does this direction of vibration represent? This direction of vibration represents nothing but the polarization. The direction of vibration of electric field vector represents the direction of polarization. So, if I again look at this wave which is propagating in positive z direction, then the non vanishing components of E and H would be Ex Ey and Hx Hy like this.

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And now if I consider a particular case, where I say let my electric field of light vibrate along x. And so, the y component is 0 because x and y are orthogonal to each other. So, so if it is vibrating along x then there is no y component.

So, electric field is vibrating along this, and the wave is propagating like this. Then such a wave is known as linearly polarized wave polarized in x direction. So, this is x polarized wave. You can understand it in a very simple way that if you tie a string, one end of the string to tie on that side of the wall. And one ~~and~~ end you keep with you and shake it, let us say this is x direction, if you shake it like this then a wave propagates like this, in the string. So, wave is going in z direction while a point on the string or particle on the string will oscillate in x direction. If you take any point on the string the direction of oscillation of that point will always ~~been~~ be in x direction. So, this is ~~text~~ x polarized wave. If you vibrate it in y direction then it goes like this. So, particle oscillates in y direction then it is y polarized wave ok.

So, if I consider this then what is the corresponding magnetic field? To get the corresponding magnetic field I use the Maxwell's equation which relates the electric field to magnetic field. Then ~~I~~ if I take the x component here Hx then del Hx over del t will be 0, if I use this. And del Hy over del t times mu would be equal to minus del Ex over del z. This gives me ~~this gives me~~ that H 0 x is equal to 0 which means that if Ex is non 0 then Hx would be 0 first thing. So, ~~so~~ if the wave is polarized along x that is electric field vector is vibrating along x, then there is no component of magnetic field vibration in x. Magnetic field vibrates along y with what amplitude it comes from here. So, if I put this Hy from here and Ex from here and simplify this then the corresponding magnetic field amplitude is k over omega mu times E 0 x, or omega epsilon over k E 0 x.

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So, in summary what are the electric and magnetic fields associated with a light beam? Well if I take the example of linearly polarized wave which is polarized in x direction, and propagating in z direction. Then the electric field associated with this is x cap E 0 e to the power i omega t minus kz it is the same equation as in the previous slide I have just generalized it instead of putting E 0 x, I have now put E 0 and x cap.

Then the corresponding magnetic field would be in y direction. So, it would be y cap H 0 e to the power i omega t minus kz. And the amplitude of H would be related to the amplitude of E by this relation. I can also write it down in terms of B this would be simplify y cap B 0 e to the power i omega t minus kz where B 0 would be nothing but k over omega E 0 because if you take this mu on this site, and combine this mu with H then it will become B.

What is k over omega? k over omega is nothing but 1 over v. So, B 0 is nothing but E 0 over v. v is the velocity of electromagnetic wave. If I consider the free space than this v is nothing but c. The velocity of light in free space, which is 3 into 10 to the power 8, which is very large. The amplitude B 0 would be much smaller than the amplitude E 0 which means that the amplitude of magnetic field is very, very small as compared to the electric field associated with light. And that is why it is the electric field that affects the retina of our eye. And so, the direction of polarization is associated with the direction of electric field vibration.

In the next lecture we would see more carefully about the polarization and we would also see how much power is associated with this kind of wave.

Thank you.