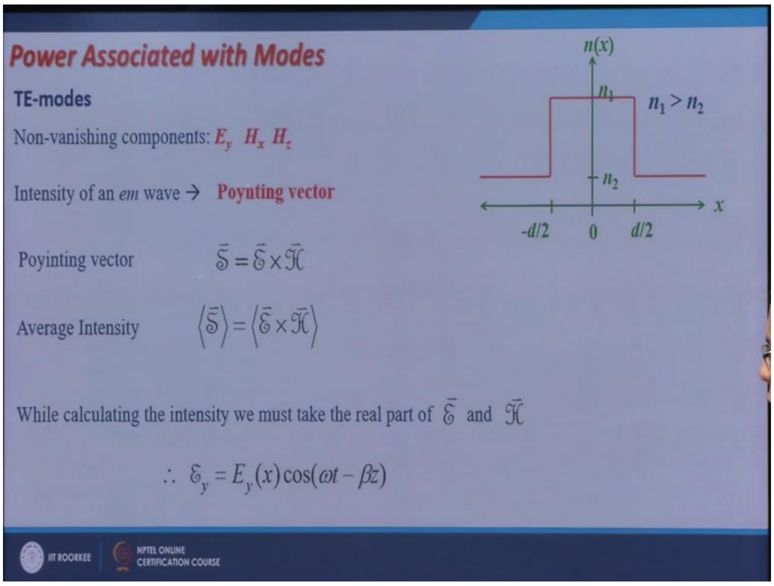
Fiber Optics
Dr. Vipul Rastogi
Department of Physics
Indian Institute of Technology, Roorkee

Lecture – 16
Electromagnetic Analysis of Waveguides- VI

After having evaluated the modal fields and propagation constants of the modes of asymmetric planar dielectric waveguide, now in this lecture let us find out how much power is associated with a mode, how much energy these modal fields carry as they propagate along the waveguide.

(Refer Slide Time: 00:51)



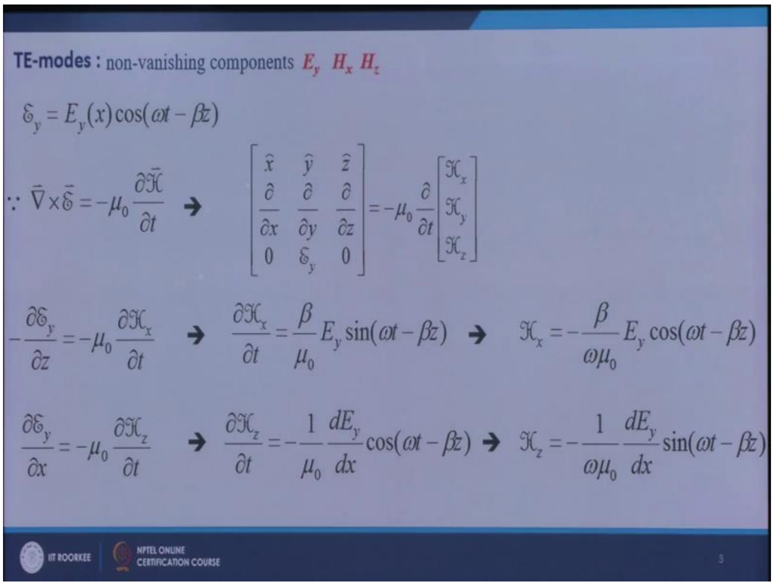
So, we will do the analysis for TE-modes we are talking about. And for this waveguide, this refractive index profile and propagation direction is z the non-vanishing components of electric and magnetic field for TE-modes are so E_y, H_x and H_z.

We know that the intensity of an EM wave is given by pointing vector. So, we now need to find out what is the pointing vector corresponding to these fields; the modal fields. The pointing vector is given by $\vec{\mathcal{S}} = \vec{\mathcal{E}} \times \vec{\mathcal{H}}$. And since we are talking about electromagnetic waves in optical frequency range, so E is fluctuating with a frequency of something like 10 to the power 15 hertz and so the magnetic field that is. Therefore, $\vec{\mathcal{S}}$ is also fluctuating at a very rapid rate. Any detector even though it is very fast detector cannot record such rapid fluctuations and so our eye. So, what we record is basically the average value- time averaged value.

So, we will find out what is the average intensity by taking the average of $\vec{\mathcal{E}} \times \vec{\mathcal{H}}$ $\langle \vec{\mathcal{S}} \rangle = \langle \vec{\mathcal{E}} \times \vec{\mathcal{H}} \rangle$ time average of $\vec{\mathcal{E}} \times \vec{\mathcal{H}}$. And since intensity is a real quantity, so while calculating intensity we must take the real parts of $\vec{\mathcal{E}} \times \vec{\mathcal{H}}$. So, for TE-modes I know that $\mathcal{E}_y = E_y(x)\cos(wt - \beta z)$. So, it is real part would be $E_y(x)\cos(wt - \beta z)$.

So, I have E_y what is left is H_x and H_z so that I can find out the pointing vector and therefore, the intensity.

(Refer Slide Time: 03:18)

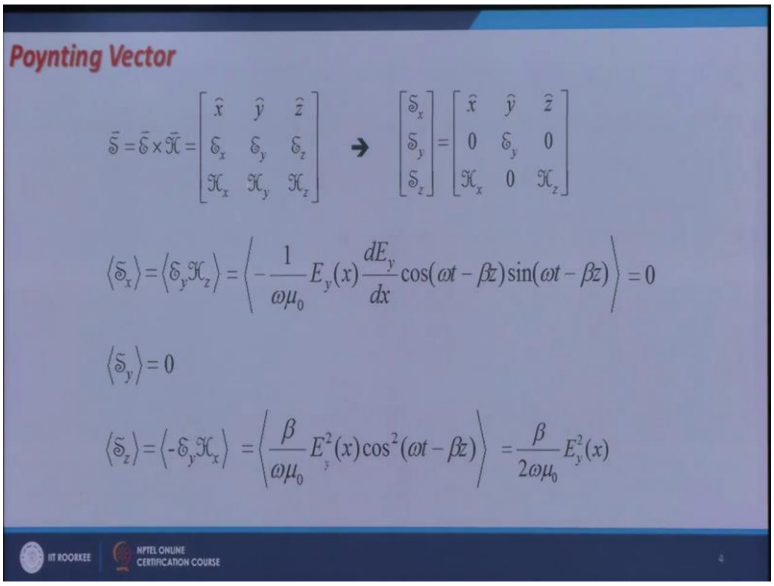


So E_y is this, how do I find out corresponding H_x and H_z? Well, I know how $\vec{\mathcal{H}}$ and $\vec{\mathcal{E}}$ are related through Maxwell's equations. It is $\vec{\nabla} \times \vec{\mathcal{E}} = -\mu_0 \frac{\partial \vec{\mathcal{H}}}{\partial t}$. So, if I expand this in matrix form it would look like this. And in $\vec{\mathcal{E}}$ I know $E_x = 0$ and $E_z = 0$ only E_y is non-vanishing for TE-modes. So, from here I can find out what is H_x, H_y and H_z. H_y is not there of course, in case of TE-modes so H_x and H_z I can get from here. So, from here I get if I take the x component $-\frac{\partial \mathcal{E}_y}{\partial z} = -\mu_0 \frac{\partial \mathcal{H}_x}{\partial t}$, and is given by this. This gives me $\frac{\partial \mathcal{H}_x}{\partial t} = \frac{\beta}{\mu_0} E_y \sin(wt - \beta z)$.

If I integrate this I can get $\mathcal{H}_x$; integrate this with respect to time so I get $\mathcal{H}_x = -\dfrac{\beta}{\omega\mu_0} E_y \cos(wt - \beta z)$. Similarly, if I take z component from here then I get $\dfrac{\partial \mathcal{E}_y}{\partial x} = -\mu_0 \dfrac{\partial \mathcal{H}_z}{\partial t}$. And if I differentiate this with respect to x. I get $\dfrac{\partial \mathcal{H}_z}{\partial t} = -\dfrac{1}{\mu_0}\dfrac{dE_y}{dx}\cos(wt - \beta z)$ integrating it over time will give me $\mathcal{H}_z$; which comes out to be $\mathcal{H}_z = -\dfrac{1}{\omega\mu_0}\dfrac{dE_y}{dx}\sin(wt - \beta z)$.

So, I have all the three components in place corresponding to TE-modes. So now, we are ready to calculate the pointing vector.

(Refer Slide Time: 05:39)



Pointing vector is given as $\vec{\mathcal{S}} = \vec{\mathcal{E}} \times \vec{\mathcal{H}} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \mathcal{E}_x & \mathcal{E}_y & \mathcal{E}_z \\ \mathcal{H}_x & \mathcal{H}_y & \mathcal{H}_z \end{bmatrix}$, so I fill in the values here now. $\mathcal{S}_z$

So, $\begin{bmatrix} \mathcal{S}_x \\ \mathcal{S}_y \\ \mathcal{S}_z \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \mathcal{E}_y & 0 \\ \mathcal{H}_x & 0 & \mathcal{H}_z \end{bmatrix}$; these are the non-vanishing components of electric and magnetic fields for TE-modes $\mathcal{E}_y$, $\mathcal{H}_x$ and $\mathcal{H}_z$. And from here I can find out what is $\mathcal{S}_x$, $\mathcal{S}_y$ and $\mathcal{S}_z$; that is x, y and z components of the pointing vector. From here I get $\mathcal{S}_x$ = $\mathcal{E}_y$. So, average value of $\mathbb{S}_x$ would be average value of $\mathcal{E}_y$, $\mathcal{H}_z$. And if I put $\mathcal{E}_y$, $\mathcal{H}_z$ as I had calculated in the previous slide so this comes out to be like this. And in this, what I see of vector

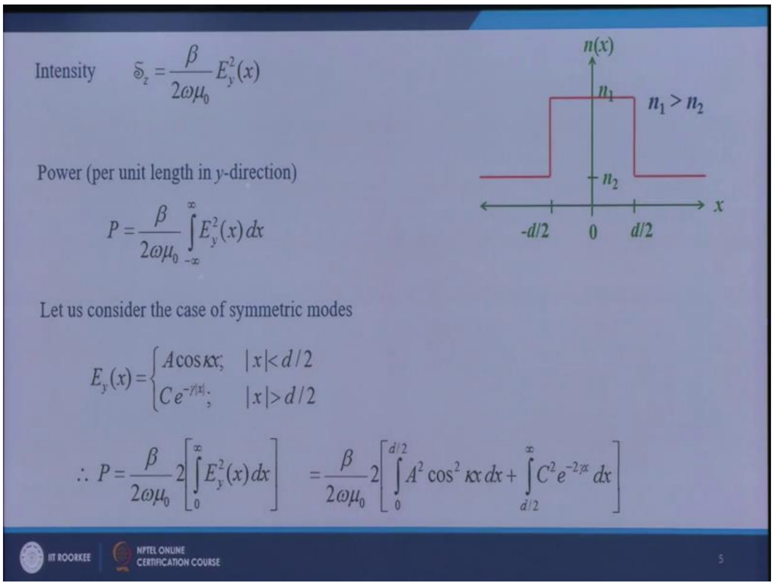
$$\langle \mathbb{S}_x \rangle = \left\langle -\frac{1}{\omega\mu_0} E_y(x) \frac{dE_y}{dx} \cos(wt - \beta z) \sin(wt - \beta z) \right\rangle = 0 \text{ ok.}$$

So, this is something like $\frac{1}{2}\sin 2(wt - \beta z)$. So, this is fluctuating and if I average it over a complete cycle for any value of *z* the average value would be 0; the time average would be 0. So, this gives me 0. So, *x*-component of pointing vector; the average value of *x*-component of pointing vector comes out to be 0.

Now let me evaluate $\mathbb{S}_y$: $\mathbb{S}_y$ is clearly 0 from here itself and $\mathbb{S}_z$ if I calculate $\mathbb{S}_z$, $\mathbb{S}_z$ will give you minus - $\mathcal{E}_y$, $\mathcal{H}_x$. So, now I substitute for $\mathcal{E}_y$ and $\mathcal{H}_x$. So, I get $\frac{\beta}{\omega\mu_0} E_y^2(x) \cos^2(wt - \beta z)$. I know for any value of *z* the time average of $\cos^2(wt - \beta z)$ would be equal to 1/2. So, this gives me $\frac{\beta}{2\omega\mu_0} E_y^2(x)$.

So, what do I see? I see that for TE-modes the average value of $\mathbb{S}_x$ and $\mathbb{S}_y$ are 0 and I get the average value of only *z*-component. And this is obvious also, this is understandable, because my mode is propagation propagating in *z*-direction so it should carry energy along *z*-direction. So, this is the intensity. So, if I know the modal field that is $E_y(x)$ then I can find out the intensity. If I integrated over the entire area then I can get the power associated with the mode.

(Refer Slide Time: 09:01)

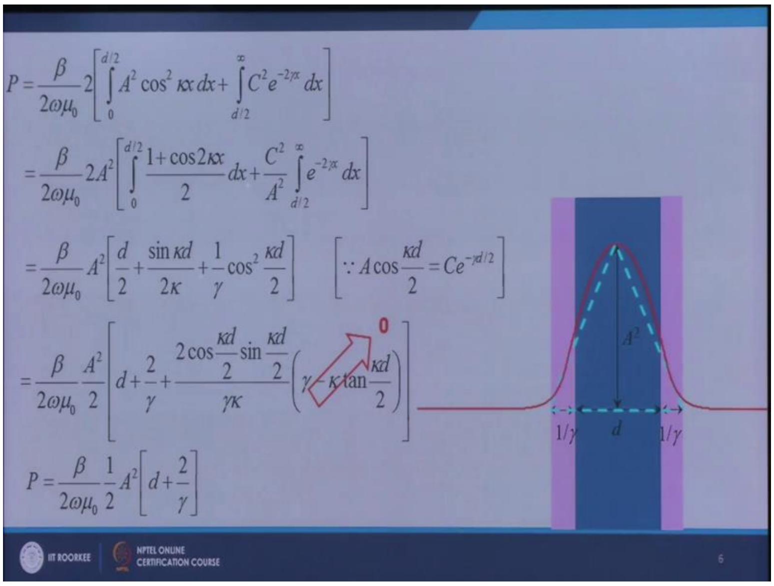


So, intensity is this, ok. To find out power I should integrate it over the area, over the transverse cross section. The mode is propagating in *z*-direction so the transverse plane is *x*-*y* plane, but why is also extended to infinity. So I cannot integrate it over *y*, I can integrate it only over *x*. So, in the case of planar waveguide I cannot have power in terms of watts, but I can have only power per unit length in *y*-direction, because I can integrate it only over *x*.

So, I get $P = \frac{\beta}{2\omega\mu_0} \int_{-\infty}^{\infty} E_y^2(x)\, dx$. And this will give me power per unit length in *y*-direction in the units of watts per meter. So, now, if I know $E_y(x)$ in the entire region I can find this out. So, let us consider the case of symmetric modes to evaluate this. The modal field for symmetric modes is given by $E_y(x) = \begin{cases} A\cos\kappa x; & |x| < d/2 \\ Ce^{-\gamma|x|}; & |x| > d/2 \end{cases}$.

So, let me substitute this into this expression. So I get, and I also make use of the fact that this is symmetric mode. So, this integral from minus infinity to plus infinity can be written as 2 times 0 to ∞. So, I make use of that and then substitute $E_y(x)$, then I get. $P = \frac{\beta}{2\omega\mu_0} 2\left[\int_0^{d/2} A^2 \cos^2 \kappa x\, dx + \int_{d/2}^{\infty} C^2 e^{-2\gamma x}\, dx\right]$.

(Refer Slide Time: 11:22)



So, let me evaluate this integral while using the boundary conditions also, because I need to relate C to A. So, here I simplify this as- I take A^2 outside and $\cos^2\kappa x$ can be written as 2 $(1+\cos 2\kappa x)/2$ and the C^2 comes out to $C^2/A^2\int_{d/2}^{\infty}e^{-2\gamma x}\,dx$.

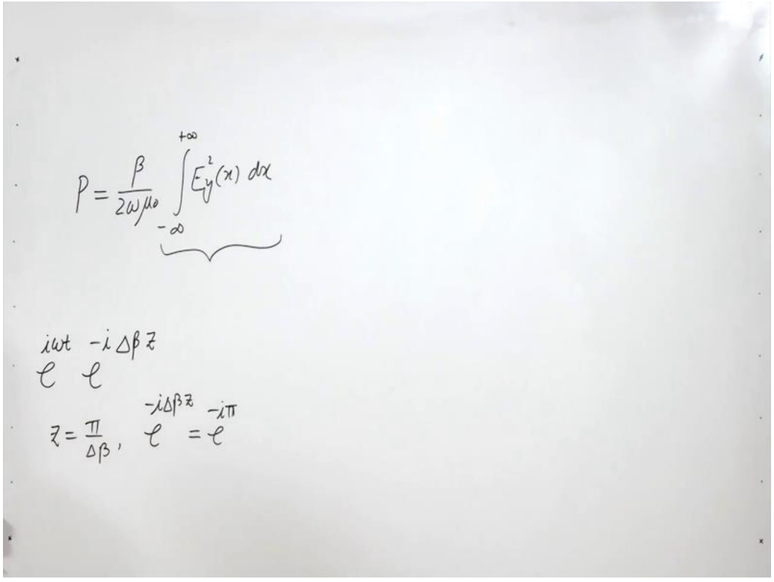
And since, from boundary conditions I know $A\cos\kappa(d/2)$ would be equal to $Ce^{-\gamma d/2}$. So, from here I will get C/A which I substitute here and evaluate this integral which is very simple $\frac{e^{-2\gamma x}}{-2\gamma}$ and then I put the limit. So, when I simplify this what I get; I get $d/2$ from here and $-\frac{\sin\kappa(d/2)}{2\kappa}$ from here and $\frac{\cos^2\kappa(d/2)}{\gamma}$ from this term. This I can further simplify. So, I take this vector to outside, so this becomes d and this becomes $\frac{2}{\gamma}$; $\cos^2\kappa(d/2)$ can be written as $1-\sin^2\kappa(d/2)$. So, this $\frac{2}{\gamma}$ which is associated with 1 comes out here and the rest of the terms I can write as $\sin\kappa(d/2)$ $\cos\kappa(d/2)$ $\sin\kappa(d/2)$, and this I take common.

So, in the bracket inside I will be left with the term which goes as $\gamma-\kappa\tan(\kappa d/2)$. It would be clear if you do this little mathematics. And why I have done in this fashion because I can see this is nothing but the transcendental equation. So this has to be 0, because

$\gamma = \kappa \tan(\kappa d / 2)$. So, if this is 0 this whole thing goes out and I get a very neat expression for power associated with the symmetric modes. And it comes out to be $P = \dfrac{\beta}{2\omega\mu_0} \int_{-\infty}^{\infty} E_y^2(x)\, dx$.

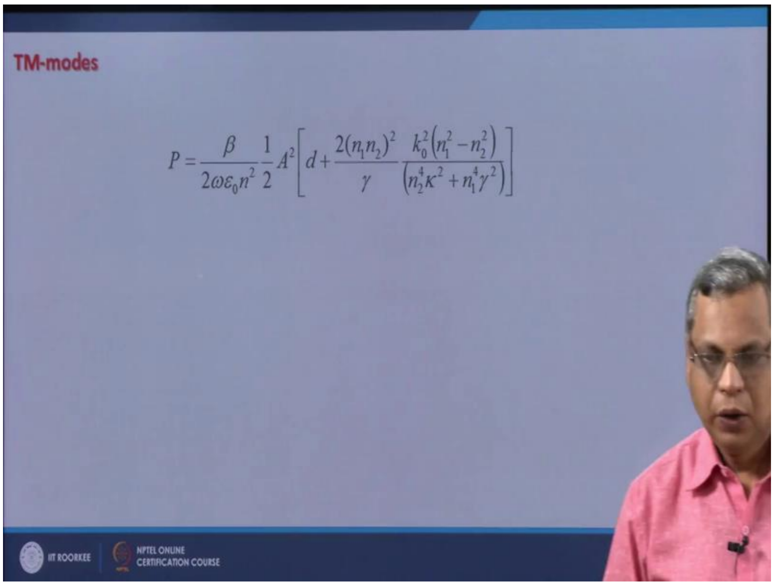
I can remember it in a very interesting way. And it is interesting to see that this term comes out to be the area of triangle under this curve. So, how you see that $P = \dfrac{\beta}{2\omega\mu_0} \int_{-\infty}^{\infty} E_y^2(x)\, dx$.

(Refer Slide Time: 14:36)



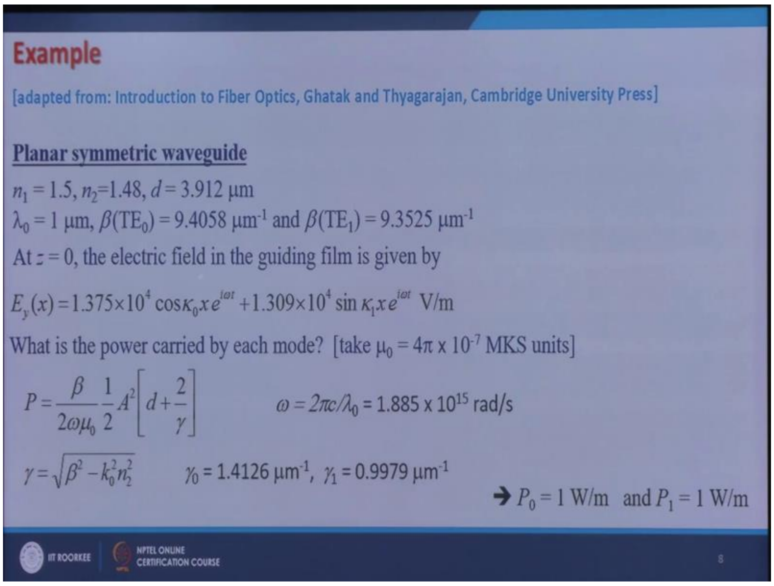
$d / 2\gamma$ So, if I find out the area under this $E_y^2(x)$ of curve then let us see what do I get. So, if this is E_y^2 as a function of x then this is nothing but A^2, this is waveguide width. And field extends into n_2 regions by distance $1/\gamma$ on either side. So, if I make a triangle which has height A^2 and base as $d + \dfrac{2}{\gamma}$, then the area of this triangle is simply $\dfrac{A^2}{2}\left(d + \dfrac{2}{\gamma}\right)$. So, this is interesting that this comes out to be like this.

(Refer Slide Time: 15:52)



For TM-modes I can do the same analysis and find out the power associated with TM-modes and it is given by this. Although, I had found out the power by taking the example of symmetric modes this expression, this expression, and this expression, these expressions are valid for antisymmetric modes also. This can be proved and this can be evaluated. So, this is how I can get power associated with the mode and these powers are in watts per meter; power per unit length in *y*-direction.

(Refer Slide Time: 16:42)



Let us work out some examples. This is the example adopted from introduction to fiber optics by Ghatak and Thyagarajan. Where I have a planar symmetric waveguide with n_1 = 1.5, n_1 =1.48, and d = 3.912 μm. At $\lambda_0 = 1\mu m$ and β for TE_0 mode is this and β for TE_1 mode is this; it supports two modes at 1 μm wavelength. If at z = 0 the electric field in the guiding film is given by this.

So, you see I have TE_0 mode and TE_1 mode, added z = 0. I excite both the modes with different amplitudes. This mode is excited with this amplitude, TE_0 mode is excited with this amplitude and TE_1 mode is excited with this amplitude. So, the total field at z = 0 is this much V/m.

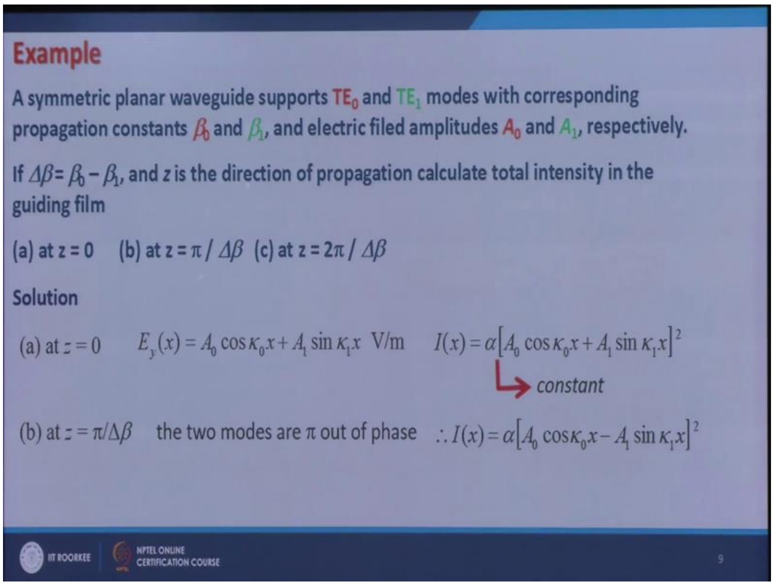
Then what is the power carried by each mode? You can take mu naught is equal to this. So, I know the power carried by a mode is given by this. So, what I need to know; I need to know what is the beta for that mode, what is the amplitude of that mode, what is γ for that mode. And of course, I need to know what is ω and ω is nothing but $2\pi C/\lambda_0$, since λ_0 is given to you so you can immediately calculate the value of ω.

A is already given to you. γ you can find out from $\gamma = \sqrt{\beta^2 - k_0^2 n_2^2}$. So, for TE_0 mode this γ comes out to be 1.4126 μm^{-1}, for TE_1 mode gamma comes out to be 0.9979 μm^{-1}. And if you put these values into this expression you will find that for TE_0 mode the power associated comes out to be 1 W/m and for TE_1 mode also you find out that the power comes out to be 1 watt per meter. In fact, these amplitudes have been adjusted in such a way that both the modes carry unity power.

When the amplitudes are adjusted in such a way then they are power normalized. If you remember that when we found out the modal fields we had retained A only and v related C to A and said that A can be found out by normalization. This is one way of normalization that you find out the value of A in such a way that that the modal field carries unity power. So, these are power normalized modes.

Let me take another interesting example of again a symmetric planar waveguide which supports two modes: TE_0 and TE_1. There propagation constants are β_0 and β_1 and electric field amplitudes are A_0 and A_1.

(Refer Slide Time: 20:33)



If the difference in their propagation constants is defined b $\Delta\beta$ and z is the direction of propagation, then what would be the total intensity in the guiding film at different values of z. First one at z = 0, second $z = \pi / \Delta\beta$, and third $z = 2\pi / \Delta\beta$. So, what do I see here at z = 0 the total field would be $A_0 \cos\kappa_0 x$, where κ_0 you can find from $\beta_0 + A_1 \sin\kappa_1 x$, where κ_1 can be found out from β_1 beta 1. So, at z = 0 this would be the total field.

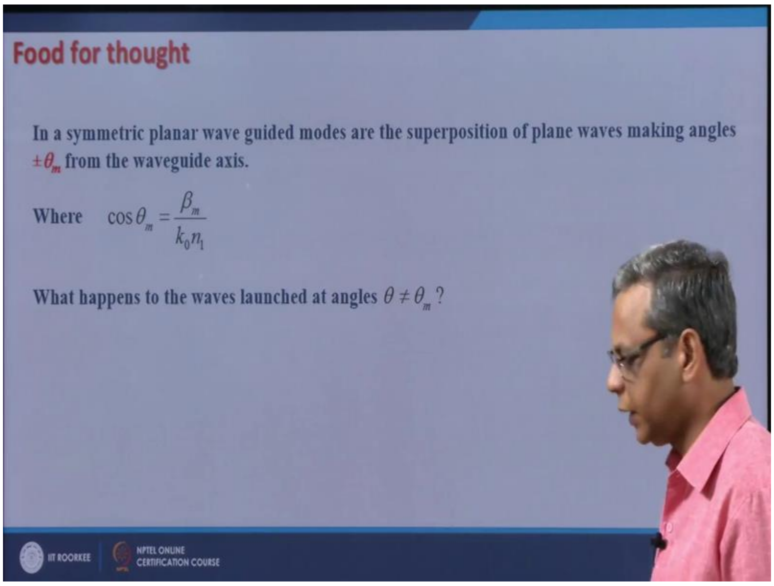
As these fields propagate it will go with propagation constant $e^{i\beta_0 z}$, this will go as $e^{i\beta_1 z}$. So, as they propagate in z-direction they would be a phase shift accumulated between them; a phase difference accumulated between them. And that phase difference would be $\Delta\beta\, z$. So, you will have $e^{i\Delta\beta\, z}$. And I know that at $z = \pi / \Delta\beta$ then this $e^{i\Delta\beta\, z}$ would simply be $e^{i\pi}$; which means that these two modes would be pi out of phase. And if we are doing in the way $e^{i\omega t}$ plus then it should be minus, so we can have this form of expression. So, instead of $+\beta z$, because I am doing it in such fashion so I can retain the same convention.

So, the thing is that the two modal fields will be pi out of phase when they traverse this distance. The intensity is nothing but the field square, so intensity would be some $\alpha E_y^2(x)$. So, this would be at z = 0, but at $z = \pi / \Delta\beta$ the total field would be this minus this, this minus this because they are π out of phase. So, the intensity would be this much. While at

$z = 2\pi / \Delta\beta$ it would be 2 pi phase shifted 2π phase shifted means there is no phase shift. So, your intensity would again be this.

And the analysis of guided modes of a planar symmetric waveguide with food for thought I had said that.

(Refer Slide Time: 24:24)



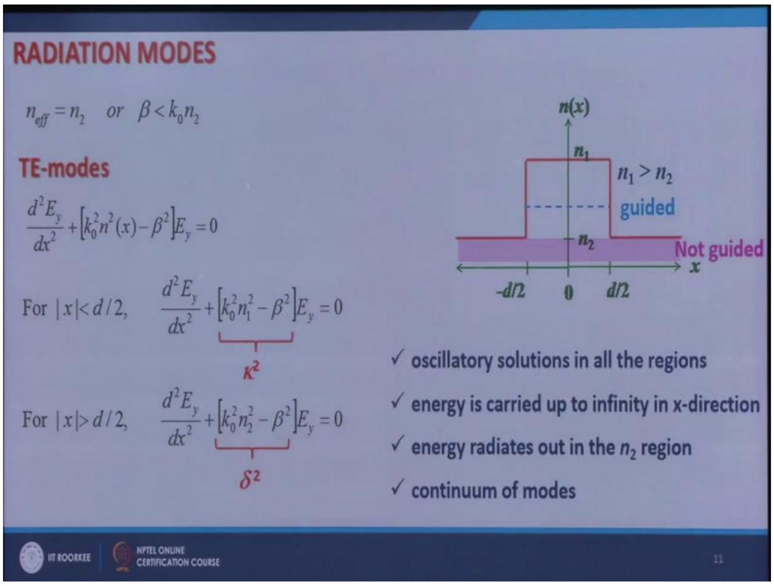
In a symmetric planar waveguide the guided modes are the superposition of plane waves making angles $\pm\theta_m$ from the waveguide axis. Where the angles are given by $\cos\theta_m = \frac{\beta_m}{k_0 n_1}$.

So, if I launched two plane waves at angles $\pm\theta_0$ then I excite the TE_0 mode; then TE_0 mode is excited and the pattern corresponding to TE_0 mode is found and it goes along the waveguide and it sustains its shape.

Similarly, if I excited $\pm\theta_1$ TE_1 mode is there. The question is what happens if the waves are launched at angles which do not correspond to these guided modes; the angles corresponding to these guided modes please think about it.

And in the end I complete this analysis of planar symmetric waveguide by briefly mentioning the radiation modes.

(Refer Slide Time: 25:41)



I have seen that if β lies between $k_0 n_2$ and $k_0 n_1$ or n_{eff} lies between n_2 and n_1 then I have guided modes. If $\beta < k_0 n_2$ they should be less than n_2. So, $n_{eff} < n_2$ or $\beta < k_0 n_2$ then what will happen. If I write down the equation wave equation for TE-modes and then I write it down in both the regions, in this region and in this region then for $|x| < d/2$. I have this equation $\beta < k_0 n_2$ and hence beta is also less than $k_0 n_1$. So, κ^2 kappa square is positive.

Now, in the region for $|x| > d/2$. I would have $\frac{d^2E_y}{dx^2} + \{k_0^2 n_2^2 - \beta^2\}E_y = 0$. And since $\beta < k_0 n_2$ then if I define this as δ^2 and δ^2 would be positive. And what I will have? I will have oscillatory solutions here as well as here. So, everywhere I will have oscillatory solutions; which means that energy is carried up to infinity in *x*-direction that is energy radiates out in the n_2 region. And these kinds of modes are known as radiation modes because the energy radiates out corresponding to these modes, and they form continuum they are not discrete modes. So, they form continuum of modes.

So, with this finish the analysis of planar waveguide, symmetric planar waveguide, whatever we have learned about the modes, modal fields, and the procedure of finding out the modes would be very useful when we will do a more complicated structure such as optical fiber.

Thank you.