Fiber Optics
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Lecture – 11
Electromagnetic Analysis of Waveguides – 1

Now, in this lecture we will start the analysis of optical waveguides using electromagnetic theory. Let us first look at types of waveguides, this is the bulk medium.

(Refer Slide Time: 00:35)

So, this is not a waveguide, when the dimensions are very large as compared to wavelength. We can have a waveguide like this where we have a channel or a material in this geometry and the dimensions are comparable to the wavelength and it is surrounded by a lower index medium like this. Then in this way I create a channel for light to flow along this. This is a rectangular channel waveguide. I can have this channel in the form of a cylinder high index refractive index cylinder which is surrounded by lower refractive index medium. Again all are comparable to wavelength the dimensions are comparable to wavelength and this is optical fiber waveguide. And then I can also have a configuration like this we are this width of the channel can be infinitely extended.

So, instead of having this kind of channel, I can have a slab thin slab of high refractive index. It is sandwiched between 2 other thick slaps of lower refractive indices. Then this is known as planner waveguide. In these 2, I have confinement in 2 dimension and propagation in this longitudinal direction. Here I have confinement only in one direction and propagation in this longitudinal direction. So, we are going to now do the electromagnetic wave analysis of such kind of media. We had seen that in infinitely extended medium where there were no such refractive index discontinuities, the solutions of the wave equations were plane waves.

(Refer Slide Time: 03:18)

If you remember that the electric field associated with that wave in an infinitely extended medium if it is infinitely extended medium.

I had seen that $\vec{\mathcal{E}} = \vec{E}_0 e^{i(\omega t - kz)}$ and $\vec{\mathcal{H}} = \vec{H}_0 e^{i(\omega t - kz)}$ and I had these vectors.

(Refer Slide Time: 03:35)

So, they have components $\vec{\mathcal{E}}_x, \vec{\mathcal{E}}_y, \vec{\mathcal{E}}_z, \vec{\mathcal{H}}_x, \vec{\mathcal{H}}_y, \vec{\mathcal{H}}_z$, this is $\vec{\mathcal{H}}_z$. And the point to be notice here was that, these E_0 and H_0 they were constants. S_0, E_0, H_0 purely constants; what we could see that in case of infinitely extended medium our direction of propagation was *z*. So, I had *z* part coming out like this and then time part coming out like this. Here also what I will do? I will take this direction which is propagation direction is *z* and *z* part would be like this itself. So, in order to now find out the propagation of light waves in such kind of medium where we have refractive index discontinuity refractive index is not uniform, when I consider Maxwell's equations in homogeneous, linear isotropic charge free current free dielectric medium.

In the previous case it was homogeneous medium now it is inhomogeneous medium. And again I write down all the Maxwell's equations Maxwell's equations go like this. But now if I look at constitutive relations, when $\vec{\mathcal{D}} = \varepsilon \vec{\mathcal{E}}$ this $\varepsilon(x,y,z) = \varepsilon n^2(x,y,z)$. So, this is not a constant scalar now, which was the case for infinitely extended medium. And again I am talking about dielectric. So, $\vec{\mathcal{B}} = \mu \vec{\mathcal{H}} \approx \mu_0 \vec{\mathcal{H}}$ because it is non-magnetic medium. So, what should I do? Know again I should form the wave equation. So, that I can find out what are the electric and magnetic field solutions. So, the usual procedure to find out the wave equation is I take the curl of this equation and use other Maxwell's equations into it.

(Refer Slide Time: 07:00)

So, I get $\vec{\nabla}(\vec{\nabla}.\vec{\mathcal{E}})-\nabla^2\vec{\mathcal{E}}=-\frac{\partial}{\partial t}(\mu\frac{\partial\vec{\mathfrak{D}}}{\partial t})$, thus in the same way as I had done in the previous case. But now here I should check that $\vec{\nabla}.\vec{\mathcal{E}}\neq 0$ I should be careful. In the previous case $\vec{\nabla}.\vec{\mathcal{E}}=0$, but now I have $\vec{\nabla}.\vec{\mathfrak{D}}$ and $\vec{\mathfrak{D}}=\varepsilon\vec{\mathcal{E}}$ depends upon *x*, *y* and *z*.

(Refer Slide Time: 07:45)

So, I cannot take that epsilon out of this del operator. So, $\vec{\nabla}.\vec{\mathcal{E}}\neq 0$ if $\vec{\nabla}.\vec{\mathcal{E}}=0$ then how this wave equation is going to change? So, let me know find out what is $\vec{\nabla}.\vec{\mathcal{E}}$ for that I take $\vec{\nabla}.\vec{\mathfrak{D}}=0 \Rightarrow \vec{\nabla}.(\varepsilon_0 n^2\vec{\mathscr{E}})=0$. So, I write it down as $\varepsilon_0(\vec{\nabla}n^2.\vec{\mathscr{E}}+n^2\vec{\nabla}.\vec{\mathscr{E}})=0$. This gives me $\vec{\nabla}.\vec{\mathscr{E}}=-\vec{\nabla}n^2.\vec{\mathscr{E}}/n^2$. I put this back into this equation and rearrange the terms to get a wave equation in this particular form.

What it is $\nabla^2\mathscr{E}+\vec{\nabla}(\frac{1}{n^2}\vec{\nabla}n^2.\vec{\mathscr{E}})-\mu_0\varepsilon_0 n^2\frac{\partial^2\vec{\mathscr{E}}}{\partial^2 t}=0$. So now, you can notice an extra term here, this is an extra term that we have got it as compared to the case of infinitely extended medium.

(Refer Slide Time: 09:16)

So, this is a wave equation in an inhomogeneous medium. What is the implication of this term now? What complicates it can introduce in our analysis? Well if I expand this then I write down this *x*, *y* and *z* components and if expand this what I see that now it is not possible for me to separate out *x*, *y* and *z*. *t* can be separated out, that is not a problem. But I cannot separate out *x*, *y* and *z*. *x*, *y* and *z* solutions cannot be separated or which I could do in case of infinitely extended medium. So, what do I do now? Well similarly if I do it for the magnetic field I will obtain an equation in *H* something like this, again there would be a middle terms which makes it impossible to separate out *x*, *y* and *z* solutions.

(Refer Slide Time: 10:27)

So now let me consider a case where refractive index where is only transfers direction which is the case of optical waveguides and optical fibers. So, so I take in general a case where n^2 is a function of *x*, *y* which can be a channel waveguide or optical fiber. So, n^2 does not vary with *z*, in this case I can separate out *z* and *t* parts. And if I can separate out *z* and *t* parts then the solution *z* and *t* solution can be written in the same way as this. So, I write down *z* and *t* solution like this. And *x*, *y* solution is still remains. So, I put it with E_0. So, the associated electric field I can write as $\vec{\mathcal{E}}(x,y,z,t)=\vec{E}_0(x,y)e^{i(\omega t-kz)}$ Similarly $\vec{\mathcal{H}}(x,y,z,t)=\vec{H}_0(x,y)e^{i(\omega t-kz)}$.

Now what I have got? I have got that the solutions here have this form. So, this is a function of *x* and *y* and this is the propagation in *z* direction. So, as if some function of *x* and *y* is propagating in *z* direction with some propagation constant β, similarly for *H*. So, these are the modes of the system. And as I will find out that there can be only certain such functions possible which sustain their shape and propagate with certain propagation constant beta these are the modes of the system. So now, my problem reduces to find out these functions, $\vec{E}_0(x,y)$ and $\vec{H}_0(x,y)$ and their corresponding propagation constants β.

(Refer Slide Time: 13:14)

Let me start with doing a very simple problem where I remove any index discontinuity even in *y*-direction. So, I take the simplest case here the variation of refractive index is only in *x*-direction.

So, I have $n^2(x)$ only and this is the case of say planner waveguide something like this, where you have this is *x*, this is *y* and this is *z*. So, where *y* is infinitely extended, *z* is infinitely extended and you have index *n* discontinuity only in *x* direction. So, here you have different refractive index here different and here different. So, $n^2(x)$ is the function of *x* only. Now if it is a function of *x* only then I can separate out *y* part also. And the solution I can write as $\vec{\mathcal{E}}(x, y, z, t) = \vec{E}_0(x)e^{i(\omega t - \gamma y - kz)}$. And *x* part will be now associated with E_0. So, E_0 is not a constant is it is a function of *x* some function of *x*. Similarly, *H* is $\vec{\mathcal{H}}(x, y, z, t) = \vec{H}_0(x)e^{i(\omega t - \gamma y - kz)}$.

So, these are the form of solutions now. What I can do? I can always choose my direction of propagation if the light is propagating in this direction I can label it as *z* or I can label it as *y* it is up to me to choose my access. So, what I do? I choose *z* axis as the direction of propagation then without loss of any generality I put $\gamma = 0$. So, you can see that if this is infinitely extended, this is infinitely extended index discontinuity is only in *x*. So, you can launch light into this in this direction and let the light propagate along *y* or you can let the light propagate along *z* you can launch it from here. So, I choose to take the direction of propagation as *z*. So, I put $\gamma = 0$.

Then I can write the solutions as $\vec{\mathcal{E}} = \vec{E}_0(x)e^{i(\omega t - kz)}$ and $\vec{\mathcal{H}} = \vec{H}_0(x)e^{i(\omega t - \gamma y - kz)}$. So, I have got similar form of solution, but it is not the same. Here E_0 and H_0 are constants; here E_0 and H_0 are the functions of *x* and these functions now I want to find out.

(Refer Slide Time: 16:58)

I need to find out how *E* varies with *x* and how *H* varies with *x*, that is what I need to know and they will give me the modes. So, I have the solutions here and do not forget that I have vector sins here which means this *E* is nothing but $\vec{\mathcal{E}} = \mathcal{E}_x \hat{x} + \mathcal{E}_y \hat{y} + \mathcal{E}_z \hat{z}$ and similarly $\vec{\mathcal{H}} = \mathcal{H}_x \hat{x} + \mathcal{H}_y \hat{y} + \mathcal{H}_z \hat{z}$. So, basically I have 6 such equations 3 in $\vec{\mathcal{E}}$ and 3 in $\vec{\mathcal{H}}$. So, if I write down the components of this then I can write them as $\vec{\mathcal{E}}_j = \vec{E}_j(x)e^{i(\omega t - kz)}$ and similarly $\vec{\mathcal{H}}_j = \vec{H}_j(x)e^{i(\omega t - kz)}$ where, *j* can be *x*, *y* or *z*.

Now, let me put these solutions into curl equations. Why I am doing this? Ultimately I want to find out how *E* varies with *x* and how *H* varies with *x*. So, I need to form a differential equation in *E* with respect to *x*. In order to do that and I know from here I will get $\vec{\nabla} \times \vec{\mathcal{E}} = -\mu_0 \dfrac{\partial \vec{\mathcal{H}}}{\partial \mathrm{t}}$ and $\vec{\nabla} \times \vec{\mathcal{H}} = \varepsilon_0 n^2 \dfrac{\partial \vec{\mathcal{H}}}{\partial \mathrm{t}}$ terms. So, that is why I put these into these equations now. When I do this then this will give me 3 equations, one corresponding to

H_x then H_y and H_z. And this will also give me 3 equations, E_x, E_y and E_z components here, let me do it. The x component from here we will come out to be if you expand this,.

So, from here you can find out 6 equations, the first one would be $i\beta E_y = i\beta E_y - i\omega\mu_0 H_x$. The x equation from here would be $i\beta H_y = i\omega\varepsilon_0 n^2(x) E_x$. The second one from here would be $-i\beta E_x - \frac{\partial E_z}{\partial x} = -i\omega\mu_0 H_y$. And here it would be $-i\beta H_x - \frac{\partial H_z}{\partial x} = -i\omega\varepsilon_0 n^2(x) E_y$.

Third one would be $\frac{\partial E_y}{\partial x} = -i\omega\mu_0 H_z$ and here it would be $\frac{\partial H_y}{\partial x} = i\omega\varepsilon_0 n^2(x) H_y$. So, I have got 6 equations which relate the electric and magnetic field components E_x, E_y and E_z and H_x, H_y and H_z. What do I do with these equations? Well, what I notice one thing that I can simplify the situation to certain extent. And how can I simplify the situation? Well if I have if I have a waveguide and I launch light into this when launching light into this I have some control on light and that is I can launch this polarization or this polarization.

This is x-axis this is y-axis. So, if I decide to launch this polarization that is $E_y \neq 0$ and $E_x = 0$ then let me see which equations do I invoke. Do invoke all the 6 equations or I invoke only a few of them?

(Refer Slide Time: 22:47)

And what I find that the equations which have are this one has $E_y \neq 0$ and $E_x = 0$. This one has $E_y \neq 0$ and this one has $E_y \neq 0$. So, so if I if I launch y-polarized wave then I invoke these 3 equations and when I launch exploitation I invoke these 3 equation. So, 3 equations can be involved at a time. So, this gives me this gives me a room to simplify the problem, because I need to now consider only 3 equations at a time. These 3 equations the blue ones are these and what I see there is they have only 3 non vanishing components of E and H and they are E_y, H_x and H_z.

In these 3 I get that there is only one component of E and that is transverse. Then these modes are also known as TE modes or transverse electric modes or transverse electric polarization. While the other 3 have non vanishing components of E and H as H_y, E_x and E_z and I see that there is only one component of magnetic field and that is transverse component then they are known as transverse magnetic modes. Or TM polarization and this will correspond to these 3 equations. So now, let me do the analysis of TE modes first. So, what I want to do? Again do not forget I want to find out how E and H vary with x and I need to find out a differential equation in E or H with respect to x. So, for TE modes I write down these 3 equations.

(Refer Slide Time: 25:13)

And what I can do now since these 3 equations relate E_y, H_x and H_z then if I know one of them then I can find out the others. So, for example, if I know E_y I can find out H_x from here and H_z from here ok.

So, what I do let me find out E_y. So, I substitute for H_x and H_z from these 2 equations into the third equation. And when I do this I form a differential equation in E_y. And this comes out to be $\frac{d^2E_y}{dx^2}+[k_0^2n^2(x)-\beta^2]E_y=0$. Where I have use the effect that $k_0=\frac{\omega}{c}=\frac{2\pi}{\lambda_0}$, where λ_0 is free space wavelength. So now, I have got now I have got a differential equation in E_y for a given $n^2(x)$. So, if I know my planner waveguide that is I know $n^2(x)$ then I can solve this equation for the given $n^2(x)$ and obtain E_y. And that will give me the modes, that will tell me how electromagnetic wave propagates in that medium of $n^2(x)$ refractive index variation. So, let me apply it to a very simple waveguide, which I call planner mirror waveguide. What is a planner mirror waveguide?

(Refer Slide Time: 27:28)

You take a very thin slab of refractive index and let us say glass. It has got a width *d* and refractive index *n*. And I polish and I sorry not polish and I deposit metal here and here. When I deposit metal on top and bottom and if I launch any light then that light will be reflected back and forth from this mirror and from this mirror and should be guided.

So, this is the simplest waveguide I can think of let me do that. So, I deposit metal on top and bottom if I look at the refractive index profile. Then I find that in this region between 0 and *d* I have refractive index n and here at the boundaries I have metal. When it boundaries I have metal then the electric field at the metal boundary should be 0 that is what I know. So, what I do know I write down the wave equation the equation which I obtained in the previous slide. And I put $n^2(x)$ as n^2 and I write it down in the region between 0 and *d*.

So, this would be the equation. So, I from here I can find out how *Ey* varies in this layer. And I know that the fields has to be 0 here. Now let me defined this $k_0^2 n^2 - \beta^2 = \kappa^2$ since I know that β^2 which is the propagation constant of the wave in this region has to be less thanit cannot be greater than $k_0 n$, because propagation constant cannot be greater than the propagation constant of the medium itself *k* of infinitely extended medium.

So, $\beta < k_0 n$. So, $\kappa^2 > 0$ which means that the solution of this equation would be $E_y = A \sin \kappa x + B \cos \kappa x$. Now my feel has to be 0 here and here. So, I apply these boundary conditions E_y = 0 at x = 0 and at $x = d$ and this gives me B = 0 and $\kappa d = m\pi$. So, since B = 0. So, this term goes off and $\kappa = m\pi / d$.

(Refer Slide Time: 30:46)

So, I put $\kappa = m\pi / d$. So, my solution now becomes $E_y = A \sin\left(\frac{m\pi x}{d}\right)$, where *m* can take integer values now what are. So, I have got $E_y(x)$ what is left corresponding β from where β are coming from here because I know $\kappa = m\pi / d$ and $k_0^2 n^2 - \beta^2 = \kappa^2$. So, this gives me that there would be only certain discrete values of β defined by β_m and given by

$$\beta_m^2 = k_0^2 n^2 - \left(\frac{m\pi}{d}\right)^2.$$

So, I have got for a planer mirror waveguide only certain functions only certain functions which has certain propagation constants they will be sustained. If I plot them then for m=1 it will look like this for m = 2 it would look like this m = 3 like this. They are nothing but if you if you look carefully they look like as the modes of vibrations of a stretched string modes of vibrations of a stretched string. So, it is similar to that. So, in the next lecture I am going to understand what do they exactly represent. I know that in a waveguide in a waveguide if I launch if I launch ray like this then it will we reflected back and forth or if I launcher wave then wave would be reflected back and forth, but how do these represent the guidance in a waveguide? So, let us understand it in the next lecture.

Thank you.