Indian Institute of Technology Madras
Present

NPTEL
NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING

TOPICS IN NON LINEAR DYNAMICS
Lecture 1
Overview

Prof. Balakrishnan
Department of Physics
Indian Institute of Technology Madras

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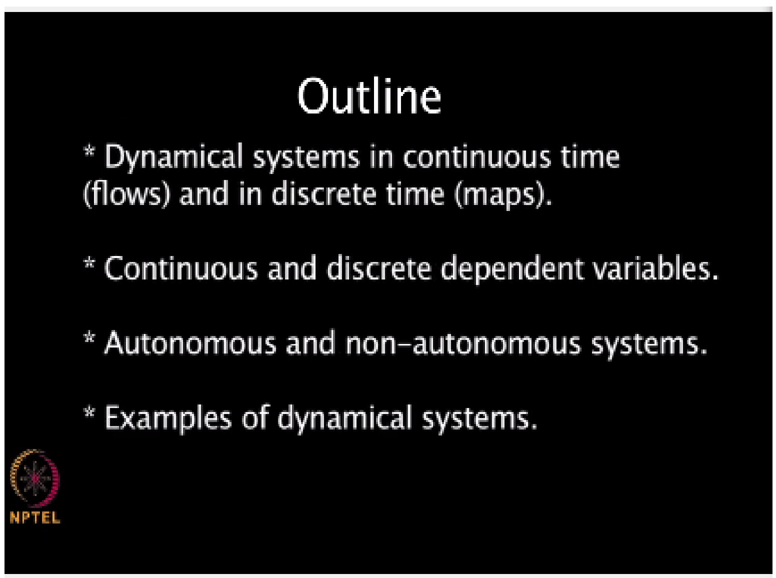

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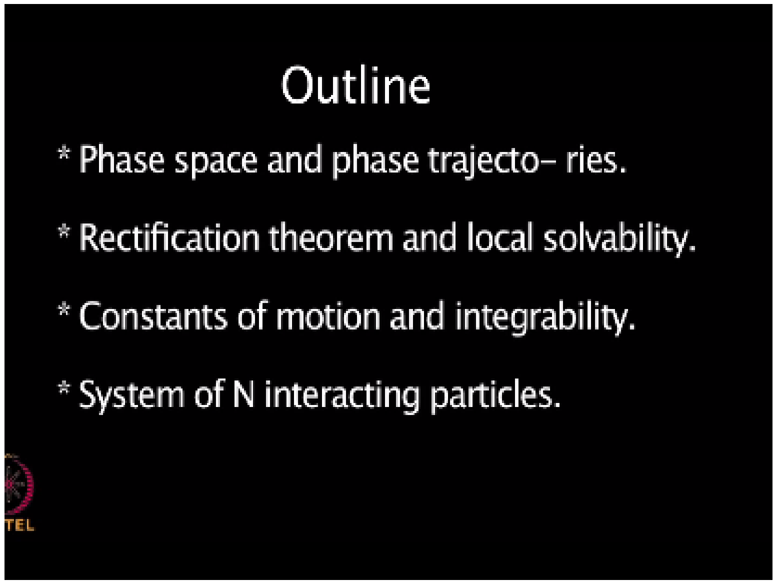

So let us begin this course on nonlinear dynamics and the title of the course is selected.

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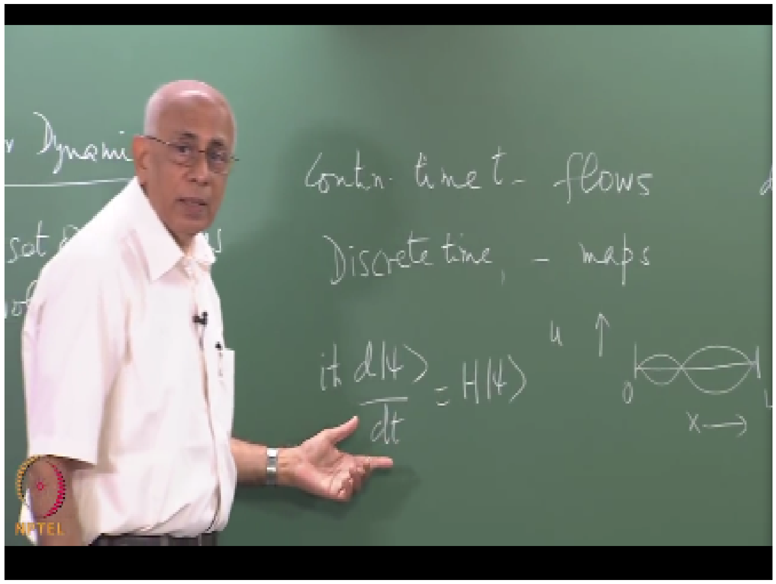

Well not even selected is just topics in nonlinear dynamics and it will deal essentially with the classical dynamics of various kinds now, I should say right away that the word nonlinear dynamics has become popular in the physical sciences for a certain kind of study of classical dynamical systems whereas in mathematics per say the word dynamical systems has a very specific rigorous connotation it is studied under something called a erotic theory and I am not going to get into those technicalities except to use some of those results.

Whenever it is relevant to whatever we are going to talk about here so I should start by asking first in the physical sciences what exactly we mean by a dynamical system as far as we are concerned anything that changes with time is a dynamical system so it is a very general definition or together and this time variable itself is something which could take on either continuous values as it normally would in Newtonian mechanics or even quantum mechanics or could take on discrete values.

So let me start by defining a dynamical system as a set of variables which change with time according to some prescribed rule so it is essentially a set of dynamical variables evolving in time now that of course is extremely general so just about everything you can think of is a dynamical system and we need to make things a little more specific so we know what we are talking about first the time variable itself well this could either be continuous-time or it could be discrete-time and let me explain.

What I mean by this if you had a set of particles moving in three-dimensional space under the influence of some potential and some mutual interaction then time is taken in Newtonian dynamics to flow inexorably from the past to the future it is a continuous time system and the system's behavior is described by perhaps a set of differential equations and the technical word for it in the case of continuous time is called a flow.

So whatever happens here we study flows in the case of discrete time for instance in population dynamics you might want to know what the populations of a set of mutually interacting species are and you might sample these populations perhaps every week or every month or every year or even every hour or every minute if it is a bacterium of some kind and the time variable then is a discrete variable as far as you are concerned I am going to use the symbol T for continuous time the standard one and for discrete time I will use a suitable variable when the time comes but this would go under named maps.

So it study flows and we study maps now what is the set of rules that we talked about what sort of variables are we talking about that to has a huge diversity for instance if you look at the position of a particle moving in three-dimensional space it is a set of three dynamical variables the position coordinates and there would also be an associated set of momentum or velocity coordinates so you would have six continuously varying dynamical variables for this so the dynamical variables could themselves be rainy Burns could be either continuous or they could be discrete the set of dynamical variables could be discrete.

Just a set of variables finite number of variables or maybe even an infinite number of variables but count ably infinite on the other hand you could have variables which are continuous take on values in a continuum so the variety is enormous and we need to be able to deal with each of these in its own way. I might as well mention that there is an even greater even more even more general possibility which is you may have a continuous infinity of variables to describe a dynamical system for instance a simple example is if you took a string extending along the x axis and it ends at 0 and L.

And you clamp it at both ends and you know ask for the transverse displacements by various points on the string but I have said some instant of time the string looks like that at another instant of time it looks like this it is vibrating instance in standing waves then if you have au here to represent the transverse displacement at any instant of time at any point it is a function of both this continuous variable X as well as the time which is continuous so it is now dependent on this system is described now by a continuum of dynamical variables if you like so the dependent variable.

In this case is U and the two continuous independent variables the dynamical equation in this case would be a partial differential equation whereas in these cases the normal cases that we talked about they would have where you have a finite number of variables for instance which vary continuously in time would be a set of differential equations you could go a little further and say even this number here this need not be a scalar if you look at Maxwell's equations for the electric and magnetic fields those are which are vector-valued fields so you have an e of (r , t) B of (r,t) you have a couple set of partial differential equations called Maxwell's equations which involve both spatial derivatives with respect.

To the position variable as well as partial derivatives with respect to time and this couple set of equations describes the dynamics of two vector valued fields the electric and magnetic fields the function of space and time so those are certainly fairly complicated partial differential equations they to constitute a dynamical system now with this huge number of possibilities we need to make things a little specific .I should mention that throughout this course we are going to do classical dynamics of course it is true that you have quantum dynamics as well and there you have a little more a general possibility.

Which is if you recall the Schrodinger equation for the state vector of a time it looks like this I over DT s H sy this is a time-dependent state vector of a system governed by a Hamiltonian whose time evolution is governed by this Hamiltonian here we have a very interesting situation where the dynamical equation is not for any physical variable not for any direct dynamical variable but rather for something abstract called the state vector from which you can deduce the probability distributions of all the dynamical observables.

That you have in the system so that is one more possibility you may have evolution equations which do not directly describe the evolution of dynamical variables but rather of their probability distribution so if there is some source of randomness in the system then you might have equations telling you the evolution of probability distributions or probability amplitudes as in the case of quantum mechanics so that two is a possibility we are not going to touch upon quantum dynamics at all in this course I am going to restrict myself to classical dynamics and that to classical dynamics in a very precise particular sense.

So to make things specific here is what we will talk about out as a dynamical variable we will assume that our dynamical system is described by a set of variables which. I will denote by X 1 X 2 up to X n n of them where n will be finite for the large part some discrete set of variables x1 to xn each of which is a function of time and this evolution is given by a set of differential equations which tell you how each of the XS vary so the equations look like this.

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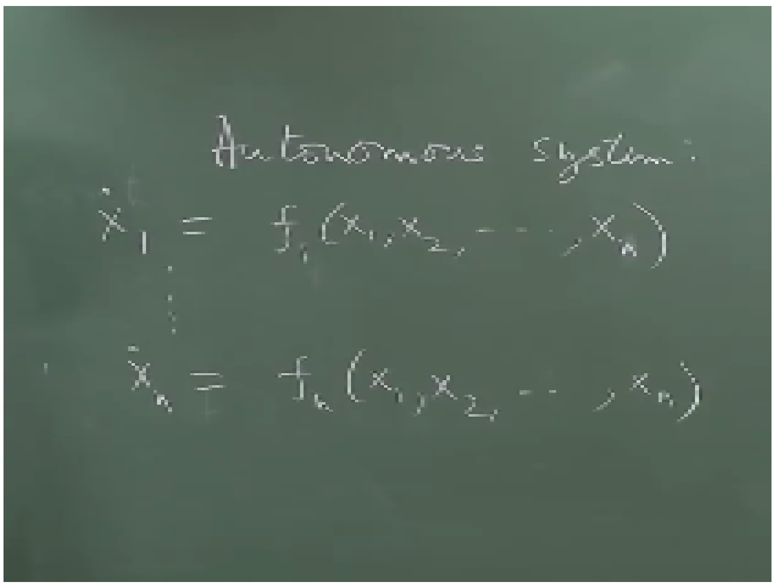

You have X 1 dot I will put a dot for the time derivative that is going to be my standard notation this is some function of all the X ones all the XS X N and possibly the time as well and I prescribe such a rule for each of the X's in my set so all the way down so X. = I need some other distinct functions so let me make it the notation simple by calling this F 1 and this is f sub n of X 1 X 2 up to xn and T such a system is going to be what I am going to call a dynamical system for most of this course the fields the functions F 1 to F n are supposed to be prescribed and we have a set of ordinary first-order differential equations for these variables.

Which are coupled to each other and which are horribly nonlinear in general which is the reason for calling this nonlinear dynamics and these functions here in general are fairly complicated but we will assume for the large part that they have a well-defined nice smooth functions with sufficient numbers of derivatives and so on this is going to be what I call a dynamical system so it is a finite dimensional dynamical system the number of variables is n and the equations are first order in time now one might ask why do we focus on first order differential equations well first of all let me spend a couple of minutes and explain very briefly why we think that why in most of the physical sciences we end up with differential equations.

To start with to describe dynamics the reason for this is that we believe there is no action at a distance in general so we believe that whatever happens to some variable at some position in space and at some given instant of time is going to be affected by and in turn is going to affect whatever happens in some infinitesimal neighborhood of it and then this disturbance or whatever can propagate and that is how the system as a whole will evolve and this idea of locality that it is always local influences that determine what happens at any particular point in space and time is what leads us to differential equations.

To start with and now why first-order well a first-order differential equation has this property in the simplest case even in a single variable you know that immediately that the solution given the differential equation the solution is uniquely determined if you specify an initial condition so this set of equations is believed to have a unique solution if you specify a given initial condition namely you specify the values of all the dynamical vary at any instant of time and then the future is supposed to be determined by this and the point is that if you have a sufficient number of variables.

If your description is complete then you are almost guaranteed that the equations going to be first order differential equations if you took this chalk to be a point particle for instance and I hold it up here and I tell you the instantaneous force on it the force of gravity due to the earth I tell you it is instantaneous position for which I need three coordinates as ng this is a point mass then the future is determined if and only if I also specify its initial velocity if I do not do that and I specify this position and the instantaneous force Newton's second law determines the acceleration of this particle if I let go it drops in a trajectory straight down.

But if I give it a little horizontal velocity it goes into an orbit I give it a little higher velocity it would go into an elliptic orbit if I give it an even higher one it would escape the earth in a hyperbolic orbit so it is clear that for this point particle you need six variables dynamical variables three of which are the coordinates at the given instant of time and the other three are the momentum or velocity components it is immediately clear that dynamics is not happening in configuration space in real space it is actually taking place.

In the full space of all the dynamical variables all the six dynamical variables in this case this is important to recognize because if I wrote Newton's equation down for a particle moving in one dimension as X double dot mass times that equal to the force acting on it f of X this might appear to be second order dynamics but it actually is not because to solve it uniquely you need to tell me both X at time 0 as well as X dot at time 0 so the correct way of writing that set of equations is to write X .= the momentum divided by the mass whereas P . / Newton's second law is equal to the force acting on the particle in this fashion so that is the set of coupled first order differential equations.

There to dynamical variables in this case X and P and each of them obey is a first-order differential equation and the fact is that you get a unique solution provided you start with initial conditions X 0 and P 0 if these are given then in principle. I can solve these two equations to determine X and P as a function of T for all time so I hope it is clear that in almost all the cases we can think of if your set of variables is complete enough the equations would be first-order equations for that matter even the Schrodinger equation for the state vector is a first-order differential equation in time.

So this conclusion is not affected by quantum mechanics it is still true even in quantum mechanics alright now that we have a set of equations of this kind let's see where we can get with it the first thing to observe is that these equations are coupled to each other and there is no guarantee at all given such a set of equations that I can eliminate n - 1 of these variables dependent variables and get an nth order differential equation for any one of them because one way to tackle this would be to say alright I have a couple set of equations let me uncouple these equations by eliminating n - 1 of these variables and writing an nth order differential equation for x1 there is no guarantee that for an arbitrary set of functions specified.'

On the right hand side you are going to be able to do this by the way I should mention here that the converse is not true for instance suppose you had an equation in one variable which looked like.

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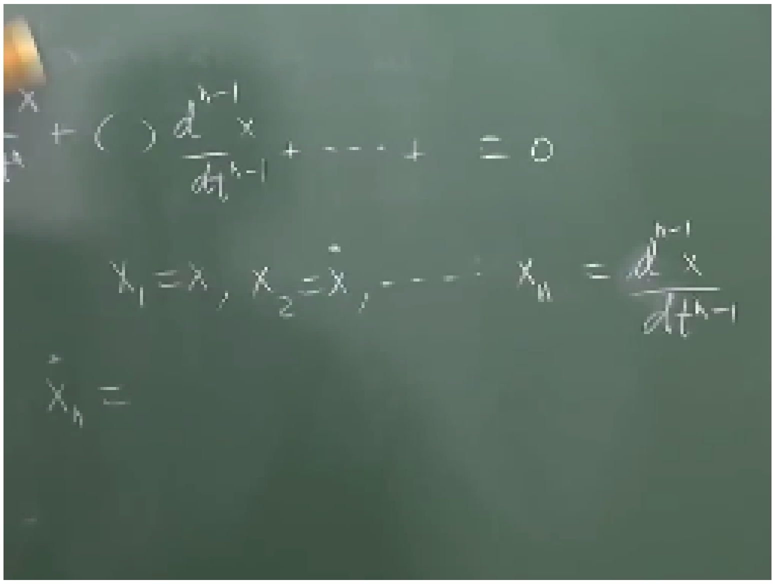

D n X over DT n + some coefficient times n - 1 derivative +etcetera equal to 0 then this equation nth order equation in a single variable X can indeed be converted to a set of n differential equations each of which is first-order by simply doing the following define x1 equal to X X 2 = X dot and so on all the way down and xn for instance now what is going to happen here is that you are going to get a set of coupled equations for the X 1 X 2 X 3 etcetera variables which would all be first-order equations the last one will involve this function that you have a non trivial thing but the others are often going to be fairly simple because the last equation for this variable would involve all the earlier dynamical variables.

So you would indeed get a set of first order differential equations right up to the last points hope you can see this xn - 1 = X is that right sorry X n is dn - 1 n -1 and then this thing here if you move this to the right hand side would give you an equation of the form X and . = a function of all the other earlier variables so here is a system of here is a single nth order differential equation which can be converted to a set of n first-order coupled equations but what I am trying to say is that the converse is not always true but we are going to stick to this as our definition of a dynamical system.

Because clearly that is much more general the second thing we notice about this set of equations is that there is a T explicit in these functions which means the rules by which these dynamical variables evolve are themselves changing with time so the system is not stationary in that sense there is an explicit T dependence and such equations are called non autonomous and the system is a non autonomous one now non autonomous systems can be far more complicated than autonomous systems and in real life very often you are faced with non autonomous systems for instance suppose you have an electrical circuit.

And this is a set of equations for the charges on various capacitors and so on and you are pumping energy into the system or you have a set of dynamical variables in any system electromechanical for instance into which you are pumping energy all the time with a time-dependent external influence or force then these equations are clearly non autonomous this case however there is a little simplification that can be made from a purely technical point of view although from the interpretation point of view it is a very different matter and that is the following if I were given a set of n non autonomous equations of this kind.

I could always define an X n +1 to be = T itself although time is not a dynamical variable dynamics happens in the arena of time as a function of time formally if I define xn + 1 = T then I could simply write these things down in terms of all n +1 variables and I have a final equation which is xn + 1 . = 1 identically and that looks like an autonomous system where there is no explicit time dependence so this looks like an n +s 1 dimensional autonomous system so to repeat an n-dimensional non autonomous dynamical system can always be made to look like an n + 1 dimensional autonomous system ,now of course the price you pay for increasing the dimensionality of a dynamical system.

Can be quite serious because something which is fairly simple if it is a two dimensional system may become extremely complicated if it is a three dimensional system which is the reason why even in ordinary simple dynamical systems such as the example of a particle moving in one dimension along the x axis which is a two dimensional dynamical system there is an equation for x and the Associated momentum P if the system is non autonomous then the equations look very different so you have X .

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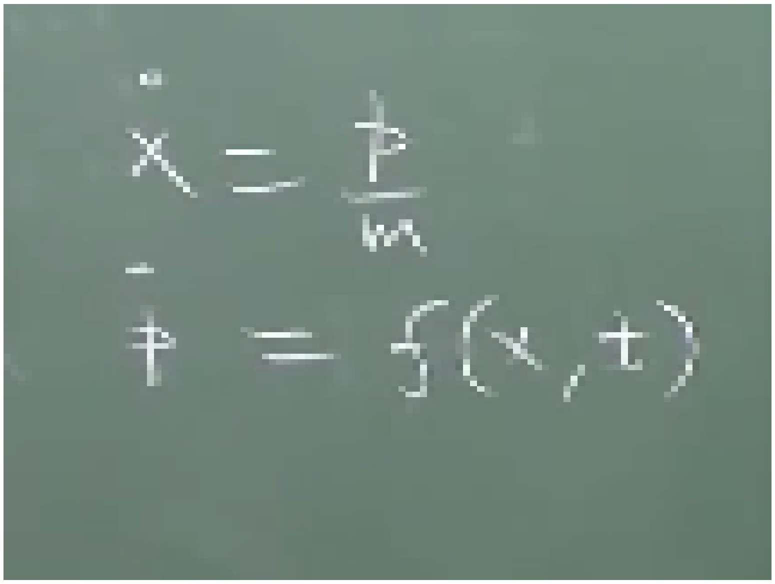

Is P over m but you have P. = a function of both (x,t) if it is in a time dependent force and then the dynamics of this system can be much more complicated than if you did not have a T dependence here for instance just to give an example if you did not have this system cannot display what is called dynamical chaos on the other hand you put in this t it can display dynamical chaos immediately because now the system has got a higher dimensional phase space.

So one has to be aware of this possibility that the dynamics can get much more complicated but the fact remains that from a technical point of view you may as well discuss only autonomous systems although if one of those dynamical variables really came by converting a non autonomous system to an autonomous one then the interpretation of the solutions has to be very carefully done we will come across examples of this later on but for the moment because of this little trick let's assume that the systems we talked about are autonomous in this sense and then a great deal of simplification occurs.

So we will look at autonomous dynamical systems in general n-dimensional it looks like that so this is going to be my basic dynamical system that we are going to talk about in the rest of this course so this is the framework in which we will now restrict our attention for the moment till we get to discrete time dynamics and maps we will restrict our attention to flows of this kind but n independent variables n of these dynamical variables dependent variables dependent time are governed by their evolution is governed by the set of dynamical equations with some prescribed functions out here.

Now I should say right away that even though these are first-order differential equations except in the simple case when these are all linear functions of the XS this set is not immediately integral you cannot write explicitly a general solution for such a set of coupled differential equations for all values of time this is not possible in general we will see why but let me say state right here that it is not possible in general on the other hand we can extract a great deal of information about the system without explicitly writing down a closed form solution for it and that is what we are going to discuss in a large fraction of this course.

What exactly what information can we get for this first we need a little bit of a compact notation so let me define this variable X to be X 1 X 2 an n-dimensional column vector if you like which has these components and similarly the natural thing to do is to define a vector valued function f which is = F 1 F 2 F n in which case this set of equations becomes simply X. = f of X autonomous so there is no extra T dependence here it stands now what do we need for the existence of a unique solution at the very least you need a set of initial conditions on all these variables so we need to know this + the initial conditions.

So you have a certain X 0 this is x1 of 0 so you have to give me an initial value of all the variables initial point in the space of these X's and then in principle the statement is the task is to now solve this different set of differential equations and extract information on the behavior of this X as a function of T so that is the task now as I said our ability to write down explicit solutions is extremely limited in very few cases such solutions can actually be found system can be integrated completely.

But a great deal can be said even without doing so provided we understand what is meant by the qualitative theory of differential equations the geometrical behavior of the qualitative behavior of the solutions of such coupled equations for that we need to be able to do this geometry we have to introduce a concept called the phase space.

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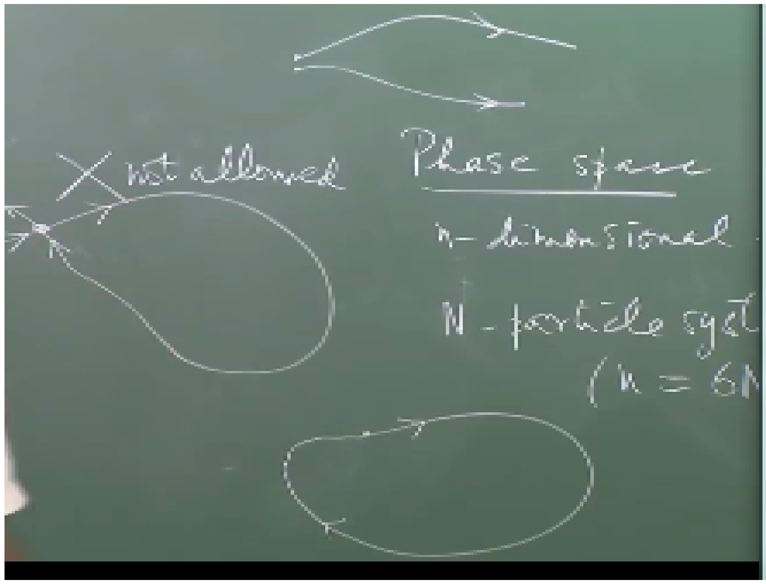

This is simply the set this is simply the space of the set of dynamical variables x1 to xn and that in our case is n-dimensional just to set a physical example if I had a single particle moving in three-dimensional space then I need three position coordinates three momentum coordinates then the phase space is six dimensional in this case if I have n different particles moving N different particles moving so here is an N particle system this would correspond to n =6 n 6for each of these particles so you have six of them n of them so therefore the phase space is six n dimensional.

If you took the gas in this room with an Avogadro number of particles the phase space dimensionality is very large astronomically large it is six times Avogadro's number if you like and that therefore makes for very complex dynamics indeed in this space in this phase space the state of this system at any point instant of time let us say T =0 is specified by a by a certain point in this space and then what happens next well what happens is as a function of time this point moves in this phase space if you solve this set of equations once you tell X tell me X of zero I tell you what is X of δ T and then X of 2 δ T and so on and then it is clear that this point moves.

In this fashion and it describes a trajectory called the phase trajectory now of course it reached this point from an earlier instant of time through the specification of exactly the same equations so in this way of looking at things making casting this as an initial value problem gives me half trajectories what happens in the future but of course this would have come from somewhere in the past to this point and then it further evolves in this direction but we are interested in the half trajectory is running from 0 to infinity unless specified otherwise there be cases where I would like to write down the full dynamics from - infinity to infinity say now there is a lot that can be said immediately just based on.

This and nothing more than that and one of them is the following for instance this phase trajectory for a given initial condition might look like this but if you had a slightly different initial condition then it could look like this it is a different phase trajectory so it is evident that for any single collection of small collection of initial States initial points values I have an set of individual trajectories which may or may not come closer together or further apart and so on and I can construct what is called a phase portrait a phase portrait will now tell me more or less what happens to all initial conditions wherever you are what is going to happen in the future well one possibility is this it could go out like that and it could come back and one might ask is this possible is such a thing possible.

In a phase trajectory and the answer is no such a self intersection is impossible for an autonomous system the reason is that you could choose any point as the initial state and we could have started with this point as the initial state and then there are two futures which goes against the theorem that you have a unique solution to these differential equations so a self intersection of this kind is not allowed so phase trajectory for an autonomous system cannot intersect itself on the other hand if this had been a non autonomous system there is no for nothing forbidding.

This because when you go out there and you come back here the rule has changed and therefore the subsequent evolution could indeed go back in this direction since the rule itself depends explicitly on time that is possible on the other hand in the cases that we are looking at autonomous systems this is not possible there is only one exception and that exception is if you start at some initial point and the system can come back to that same initial point and then thereafter continue forever in the same orbit the reason being that this set of equations is explicitly independent of time.

So therefore if you started at this point and it came back to this point it is doomed to continue the same path forever afterwards and what does this mean at this point all the variables dynamical variables have returned to precisely their starting values which means the motion is periodic so this immediately implies that a closed phase trajectory implies and is implied by periodic motion that is an important elementary but important conclusion because we certainly should like to know be able to identify those parts of phase space where the motion is periodic as opposed to those where it's not periodic.

So this is an important observation that a closed phase trajectory in the phase space the full phase space is periodic motion it is not enough for some of the variables to come back to the starting points for instance if I took a simple pendulum ,if I go out to the end of the amplitude and come back it is come back to it is starting point as far as the amplitude as far as the position is concerned but not as far as the velocity is concerned because initially the velocity is directed to this side but when it comes back here it is directed to this side so it needs to come back all the way and that completes a full orbit.

In phase space and then you have periodic motion so that is important to recognize the next point is what in principle is stopping me from solving these equations what is stopping me at all the answer is the following it is interesting and important let us take a look at these equations again.

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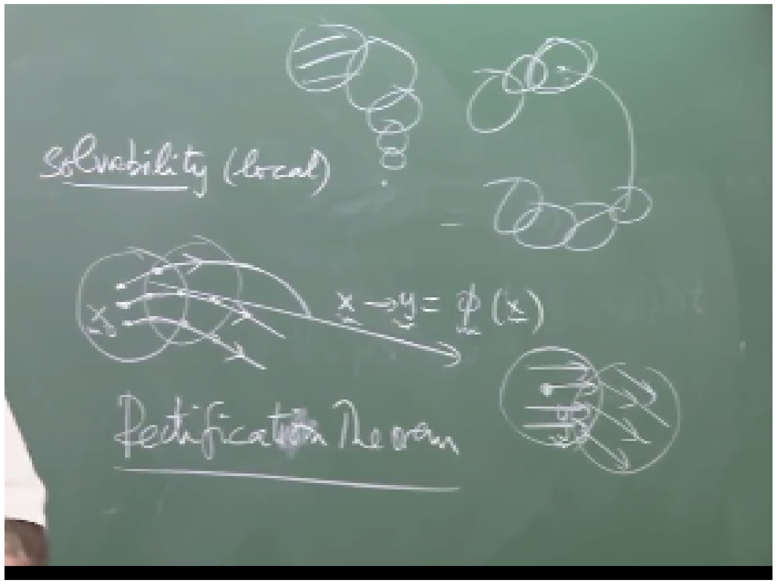

If I tell you X at time 0 I can tell you what X is at time δ T that straightforward because all I have to do is to use the value of whatever these functions are at the initial instant of time multiplied by δ T because remember these are first order differential equations here so this is = X at 0 +F at X 0 so let us call it X let me call it X 0 X and 0 δ . I have just taken this differential equation kept things to first order in this infinitesimal time δ T and that is it so if I start at some point in phase space a little time δ T later I am there so I know this and if I make δ T fine enough I should be able to draw complete a continuous curve and call it the phase trajectory and once.

I am here I did say play the same trick again and I move forward and this method is like the method of isoclines you build up a phase trajectory in this fashion so it is clear that there is no difficulty in principle to numerical solver ability of these equations and then I could go to a slightly different initial condition here and play the same trick and compute step by step by step in eternal step. I compute what the phase trajectory looks like so certainly local solvability local infinitesimal step at a time is not difficult in principle it is straightforward to find local solutions for any given for almost all given initial conditions.

That is not the difficult part now the way to look at this is to understand that the velocity of this representative point in phase space is given by this vector field this F is a vector field and it gives you the local phase space velocity now once you specify that local solvability is not difficult and what is being plotted here are really the field lines of that vector field and this is always guaranteed local solvability in fact it is codified in a formal theorem in calculus multivariable calculus called the rectification theorem and it says the following it says if in the space of excess at any given point you give me.

This vector field whose field lines look like that I can always make a change of variables a smooth change of variables from X to some Y which is some function of X such that this point X 0 gets mapped on to some point Y0 here and more important than that these field lines which look curved at this point they get mapped on this infinitesimal neighborhood gets mapped on here such that the field lines look like this and that is why it is called the rectification theorem.

Because they look like straight lines parallel to each other now what it means in this language is that under this change of variables under this change of variables the new set of equations in the variables y - y1 2 y n looks very simple looks like Y 1 .=1 in suitable units y2 dot is 0 all the way Y and . = 0 and what are the solutions of this is simplicity itself well it says y1 of T is y1 0 +T and Y 2 y 3 up to yn are constants they do not change with time at all so that is why the field lines go parallel to each other it is only the one direction that you have any change of dynamical variables all the others is supposed.

To remain constant here and if you like the vector field has been rectified so this is always possible by a rigorous theorem which says that this equation can always be solved numerically solved if you like locally so now if I want to go beyond this region then, I must start with this as the initial point and in this infinitesimal neighborhood. I must apply the rectification theorem again which would perhaps mean that you now have a new vector field which looks like this in this case and so on so infinitesimal interval by interval I can now solve this equation locally why is it that this does not imply that I can once and for all solve the set of equations.

And right once you give me some initial value X of 0 given why is it that I cannot solve the set of equations and write once and for all X of T equal to some known function which I can write down explicitly of X of 0 and T if I were able to do this then for all T I have an explicit solution to this whenever I can do this I say the system is integral there is different kinds of integrity but the one that I will talk about here is simply in this sense that if I can write an explicit time dependent solution valid for all T I say the system is integral the very few systems which are integral we will come to that but I want you to appreciate.

The fact and this is what I am going to emphasize that local solvability does not imply inerrability so solvability does not imply the other way it is true if you give me if you tell me that in a given system I know the solution explicitly for all T then of course you know it locally for infinitesimal changes in t see but the going the other way is not directly possible in most cases and the reason for it is technical there are several possibilities the most common ones are the following these map functions Л that I write down to go from one to the other by the way.

I should have used some other notation here let us call it some other sine of X ,T it is not the map function here this map function here could be such that my initial neighborhood out here the next time it does this and then it does this and then it does this and this can Peter out to a point so this mapping may be valid in smaller and smaller neighborhoods till it kind of disappears shrinks to a point in which case the method is useless after that there is another possibility and that is, I start at some point here and they continue this sequential mapping here and I come back to the original point to this point here and I find a different map which means that there is no single valueless anymore.

So both these possibilities happen and you can sort of guess that in this case there must be some kind of singularity in this vector field somewhere which is preventing this from being you solved globally everywhere here so there are other possibilities as well but without going into that let me just state as a matter of fact that inerrability is a very rare event in dynamical systems it does happen in some very important examples we are going to talk about criteria for integral ax t for instance for Hamiltonian systems and so on but it is a rare event now let me go on right here and explain why it is such a rare thing what is so difficult about this in slightly different terms what is it that we need.

To integrate let us go back and ask for about the meaning of integration from elementary calculus you take any of these equation and you integrate it there is an integration constant so when you integrate n such equations you are going to have n integration constants whose numerical values will be determined by the initial conditions so it is clear that there are certain constants of the motion which are determining what is happening to the phase trajectory for instance suppose it turned out.

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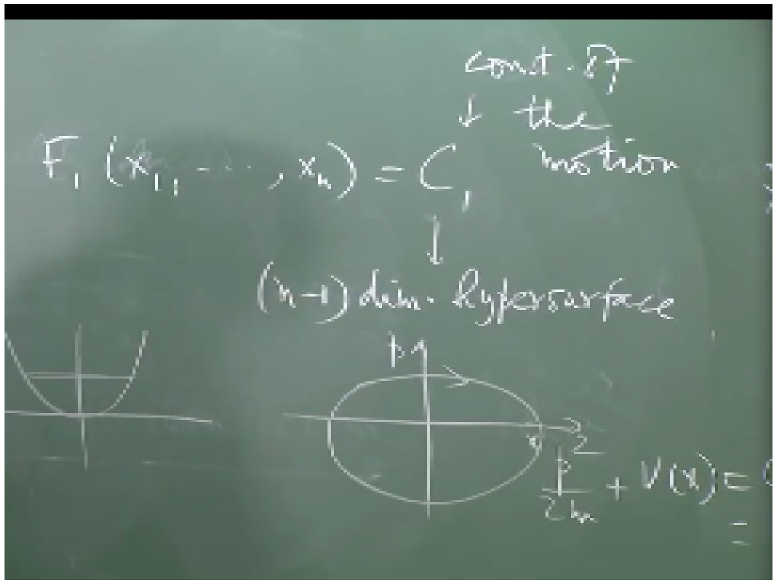

That some given function f 1 not this function some other given function f 1 of all the XS equal to a constant let us call it C 1 during this motion I discover that this particular combination is a constant for instance a particle moving in a one-dimensional potential which is a conservative system no friction or anything I know the total energy is constant the total energy is a function of the velocity as well as the position kinetic +potential and that is a constant of the motion so this thing here is a constant of the motion what does that imply for the phase trajectory.

It implies that this combination has always got to be constant and this defines for you in this n dimensional phase space and n - 1 dimensional hyper surface for instance in the case of the particle moving in a potential if this is what the potential looks like and the particle moves back and forth in one dimension we know that the sum of the potential and kinetic energy is constant the motion is periodic and the phase trajectory looks something like this now what is this thing here it is an equation which says P^2 2m +V of X = a constant this is the x-axis and that is the p axis and this constant has the physical meaning =of the total energy of the system.

So the system is now constrained if you like restricted to moving on this surface quote-unquote in this case it is just a curve so every time you have a constant of the motion the phase space available to this system is decreased by one dimension and you have an N -11dimensional hyper surface if I discover the second constant of the motion so let us write an independent second constant of the motion.

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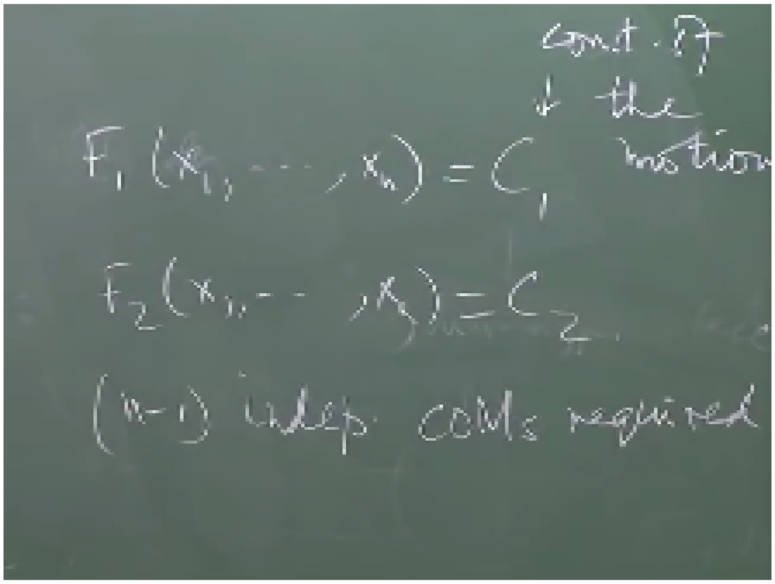

F 2 of X 1 to xn = C 2independent of C 1 these are functionally independent quantities here then this two restricts your motion to some other n minus 1 dimensional hyper surface and this representative point must move on this surface as well as this in other words it must move on the intersection of these two surfaces but that is n - 2 dimensional because every time you put one more constraint the dimensionality of whatever is available to you drops by 1 therefore the phase trajectory itself it is clear is the intersection being a one-dimensional object is the intersection of n- 1 such surfaces hyper surfaces in other words for you to integrate completely you need to be able to write down n - 1 functionally independent constants of the motion.

If you could do that then the mutual intersection of these n -1 constants of the motion already fixes your face trajectory it also tells you why the phase trajectory can be exceedingly complicated because it is clear that the intersections of surfaces even if they themselves are quite smooth can become quite intricate when you have more and more of them and in principle the phase trajectory in n dimensional phase space can be exceedingly complicated looking but not self intersecting in for any initial condition and you need to be able to write it down explicitly in aristocratic and inconstant of the motion.

But you do not have this because constants of the motion are very rare to come by let me anticipate what we are going to say later by saying that the existence of a constant of the motion implies in general a certain symmetry in these equations so the more symmetry you have the more likely it is that there are constants of the motion but if you took an arbitrary generic system there is no reason why there is any special symmetry then you may not have constants of the motion available to you so that is one way of looking at why integrability is very rare unlike local solvability just very common but integrability is rare.

Let me end by giving the same example of n particles moving in space and then we will come back to this just to show you what happens in that case.

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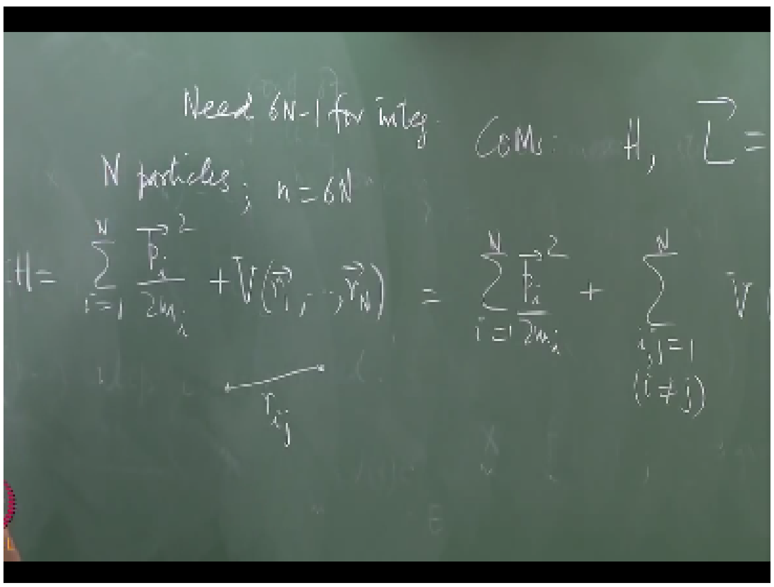

So let us look at a system of n particles for which the phase space is 6 n dimensional therefore I need 6 n - 1 independent constants of the motion to be able to integrate this completely but then what happens is that the system is described by an energy function called the Hamiltonian which looks like summation I = 1 to n Pi^2 over twice Mi but is the mass of the particle +a potential if there is no external force and the particles are mutually interacting with each other which is a function in general of all the coordinates of the particle that is a general system we will make it more specific by saying oh the interaction is always pair wise.

These particles interact to at a time with each other and depends only on the distance between the two particles which is what happens in most cases so if you had a particle I here and a particle J here the distance (r i j) is all that this interaction would depend on then this Hamiltonian simplifies a little bit and this becomes summation VI ^{2} over 2m I I = 1 to n + a summation over these pair wise potentials each of which is a function only of the distance between two particles ij summed over from I J = 1 to N and you do not want self interactions of these particles.

So I not = J that is what the total energy function or the Hamiltonian of a conservative system of n particles interacting by pair wise interactions which are central forces this is what it looks like now what is the symmetry of this Hamiltonian here what is the number of constants of the motion that we can write down here well we know from elementary mechanics that the Hamiltonian itself for this non autonomous for this autonomous system it does not explicitly depend on time it is a constant of the motion and the value of the Hamiltonian is the total energy of the system.

So the constants of the motion of the system are the Hamiltonian itself which gives the total energy what other function of the dynamical variables can we think of which are constants of the motion which do not change as a function of time although the RS and the P is all change as a function of time what are the constant can we think of well this whole system is rotationally invariant I can make a sum chosen axis. I can always orient it as I please and this the Hamiltonian does not change at all another way of saying it is that the total angular momentum is a constant in this case so it is clear that this quantity L.

Which is a summation I = 1 to N or I cross P I that is a constant of the motion it is guaranteed that the total angular momentum of the system about the origin is a constant of the motion there are actually 3 constants of the motion here because it is a vector function what else is constant well there is no external force.

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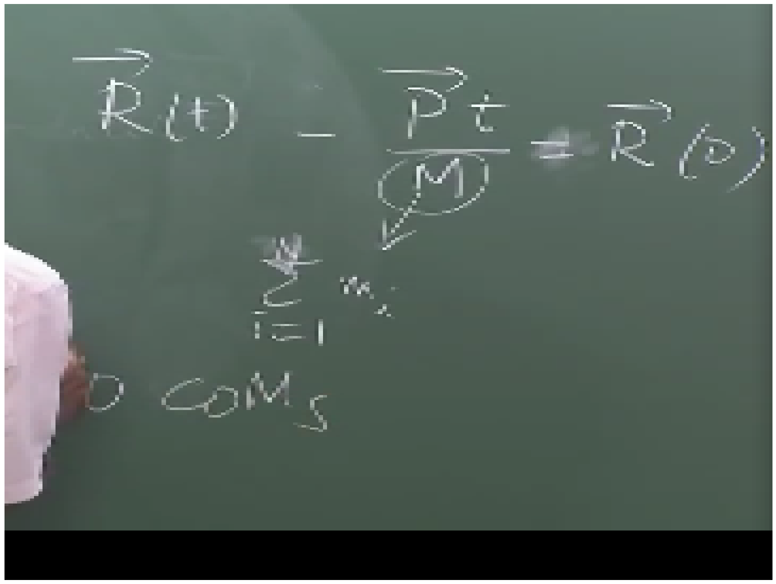

On the system so the generalization of Newton's third law is applicable here and it says that the total linear momentum of the system P = summation I = 1 to N P I is a constant of the motion so now you have three of these quantities here because a 2 is a constant of the motion so we have 1 +3 4 +3 7 constants of the motion okay, what else is a constant of the motion well it is clear that in this case since there is no external force on the system the center of mass of the system must either be at rest or be moving with constant velocity given by P over the total mass of the system .

So it is clear that this quantity R which is a function of time by the way = P which is also constant of the motion so might as well write it as P T over m this stands for summation over I = 1 to N M sub I + at zero so the combination R - P of T P times T over m = this is a constant of the motion there are three of those constants of the motion but their time-dependent and that is nothing strange because the time dependence in this quantity will cancel the time dependence due to this and make that a constant of the motion here so those are three more constants of the motion by the way these are called Galilean invariance Galilean constants of the motion.

And for a general potential about which you do not know anything else that is it there are no more constants of the motion so the total number that we have here ten constants of the motion these are ten constants of the motion whereas the number that you need for integrability you need 6 N-1 for integral and that is pathetically small compared to what this number is for any reasonable n forget about Avogadro's number even for n = 3 it already becomes impossible so in general even the three-body problem.

In this case is not integral in that sense this particular kind of Hamiltonian forget about more complex systems so the one and two body problems are indeed integral as we can show but the three body problem is not integral here so already in Newtonian mechanics you see the seeds of complexity here in fact such a system will in general be chaotic as we will see and the reason is that you do not have a sufficient number of constants of the motion and these are going to play a fundamental role the constants of the motion and when we come to Hamiltonian dynamics.

I will explain what significance of this these constants of the motion are but this is the reason in a nutshell general dynamical system is solvable but not integrable because you do not have sufficient symmetry for the existence of a sufficient number of constants of the motion to integrate the system X + so we will take it from this point next time you.

Online Video Editing /Post Production

K.R.Mahendra Babu
Soju Francis
S.Pradeepa
S.Subash

Camera

Selvam
Robert Joseph
Karthikeyan
Ram Kumar
Ramganesh
Sathiaraj

Studio Assistants

Krishankumar
Linuselvan
Saranraj

Animations

Anushree Santhosh
Pradeep Valan .S.L

NPTEL Web &Faculty Assistance Team

Allen Jacob Dinesh

Bharathi Balaji
Deepa Venkatraman
Dianis Bertin
Gayathri
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Jason Prasad
Jayanthi
Kamala Ramakrishnan
Lakshmi Priya
Malarvizhi
Manikandasivam
Mohana Sundari
Muthu Kumaran
Naveen Kumar
Palani
Salomi
Senthil
Sridharan
Suriyakumari

Administrative Assistant

Janakiraman.K.S

Video Producers

K.R. Ravindranath
Kannan Krishnamurty

IIT Madras Production

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