Kinematics of Machines

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Module - 2 Lecture - 2

The topic of today's lecture is number synthesis. During this stage of kinematic synthesis called number synthesis, we determine the type and number of different types of links and the number of simple pairs like revolute or prismatic pairs that needed to yield a single degree of freedom planar linkage. It is needless to say that all this single degree of freedom planar linkages will satisfy the Grubler's criterions which are discussed earlier. However, before we get into the discussion or details of number synthesis, we shall first prove certain basic results, which are of vital important for number synthesis. The first of these two questions are, what is the minimum number of binary links that such a linkage must possess?

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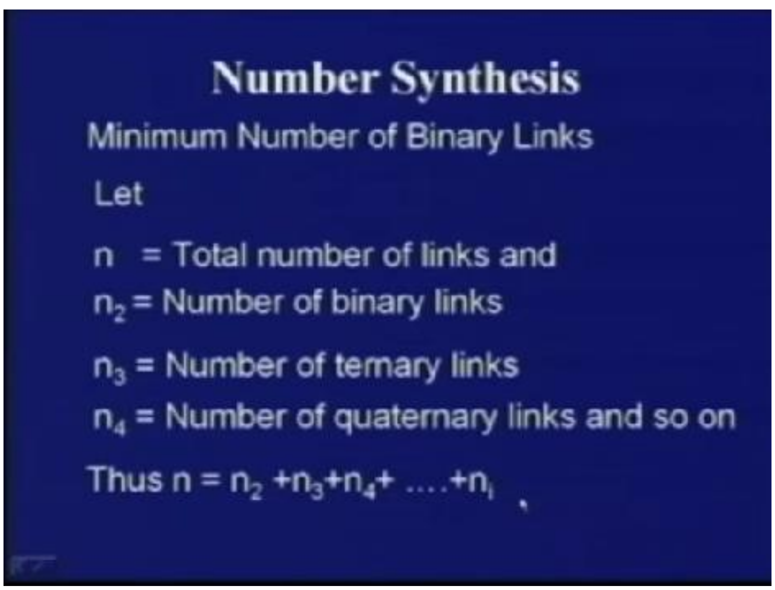

So, we determine the minimum number of binary links in a single degree of a freedom planar linkage. Let 'n' be the total number of links in the linkage, 'n_2' be the number of binary links, 'n_3' be the number of ternary links and 'n_4' be the number of quaternary links and so on. Thus we have,

the total number of links, $n = n_2 + n_3 + n_4 + \ldots + n_i$

where i denotes the highest order link that is present in this linkage. So, our first task is to determine the minimum value of n_2. Towards this goal let us consider this figure.

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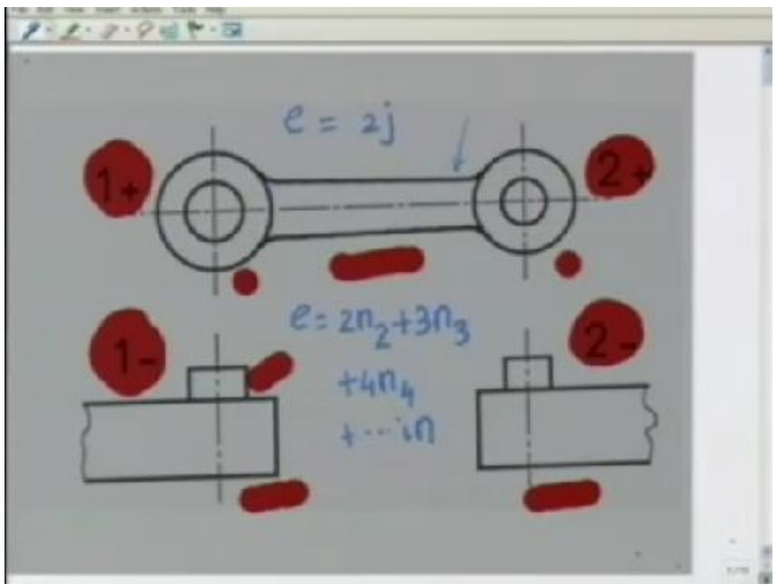

In this figure, we see there is one link which is connected to two other links through the revolute pairs here and here. To note that at each of these revolute pairs we have two elements, say 1– which is the pin which goes into the hole which is denoted by 1+. So this 1+ and 1–, we shall call elements, thus at each revolute pair we have two elements. Similarly, the two elements at the other revolute pairs are 2+ and 2–.

In this way, if we connect count the total number of elements that I can write 'e' should be equal to twice the number of joints or pairs say that is e = 2j. We can also count this number of elements from this links. This is a binary link which has two elements because it is connected to two other links, two revolute pairs. Similarly, a ternary link, we have three elements because it is connected to three other links and a quaternary link we will have four such elements. So if we count the total number of elements from the view point of links then I can write, $e = 2n_2$ (where n_2 is the number of binary links) + $3n_3$ (where n_3 is the number of ternary links) + $4n_4$ (where n_4 is the number of quaternary links) + …. + in_i (where n_i is the number of ith order link). So we have just now seen that the total number of elements can be counted from two viewpoints.

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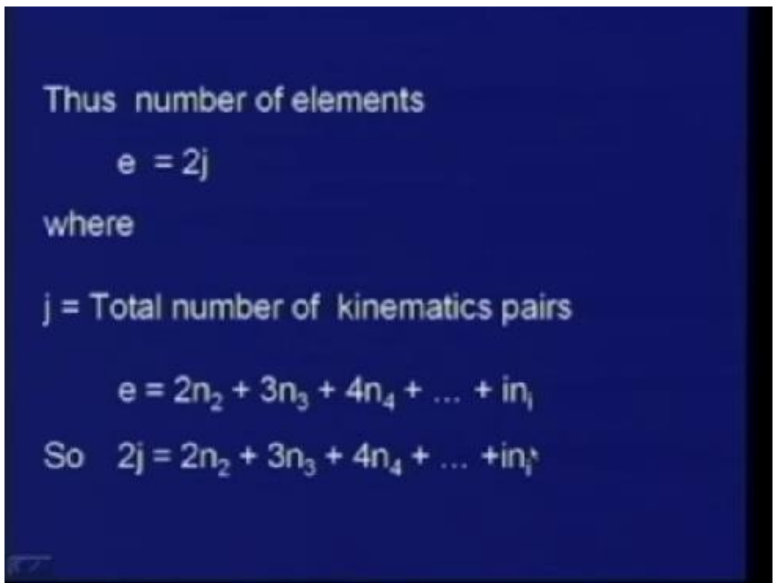

If we count it from the viewpoint of number of pairs then I can write the total number of elements $e = 2j$. But, if we count the number of elements from the viewpoint of different links or of different orders then we can write the total number of elements $e = 2n_2 + 3n_3 + 4n_4 + …. + in_i$. We can equate these two numbers of elements counted from the viewpoints the kinematic pairs and from the viewpoint of different order links, we can write,

$$2j = 2n_2 + 3n_3 + 4n_4 + …. + in_i.$$

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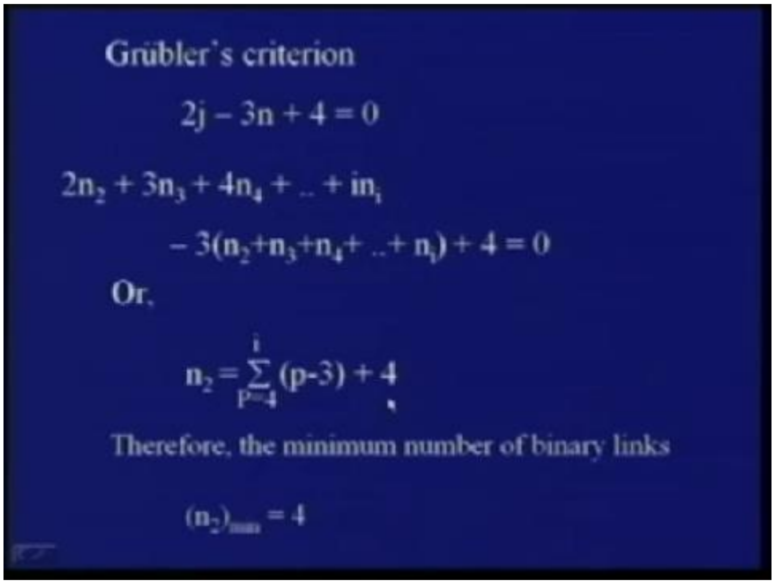

As already told, that all these linkages must satisfy the Grubler's criterion which is, $2j - 3n + 4 = 0$, where 'j' denotes the number of kinematic pairs and 'n' denotes the number of total links. Now substituting e = 2j, which is just now saying to be given by,

$$2n_2 + 3n_3 + 4n_4 + \ldots. + in_i - 3(n_2 + n_3 + n_4 + \ldots. + n_i) + 4 = 0$$

Simplifying this equation we get,

$$n_2 = \sum_{p=4}^{i} (p-3) + 4$$

where this 'p' denotes the number of quaternary links and higher order links.

So we can easily see if the sum is 0, the minimum number of binary links, $(n_2)_{min} = 4$. This again convinces us what we have seen earlier, that this simplest linkage must have 4 binary links, what we call four bar linkage.

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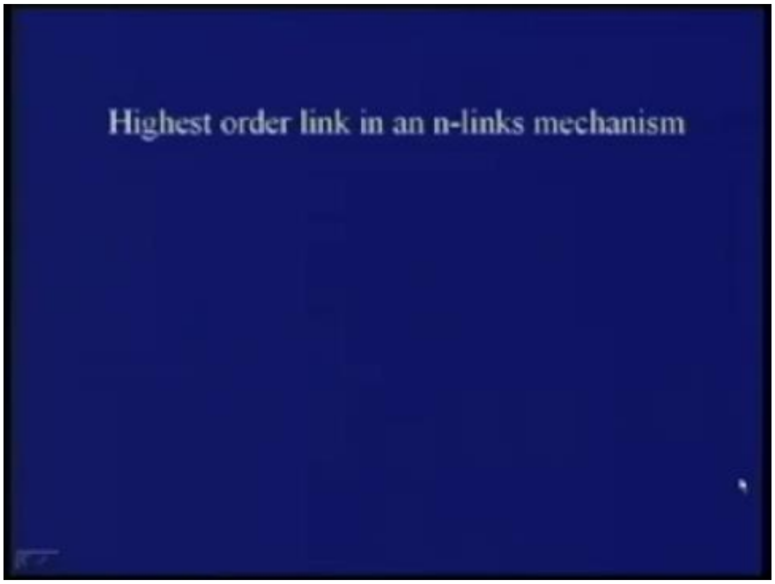

Next, we would like to add another question that is what is the highest order link in an ‘n’ link mechanism? That means, the total number of links is ‘n’ then in such a linkage what is the highest order link? We should try to answer this question in a reverse manner. We will say, the highest order link be ith order that is we have some n_i’s. Then what is the minimum number of links that is needed to produce the single degree of freedom planar linkage. Towards this goal let me consider the following figure.

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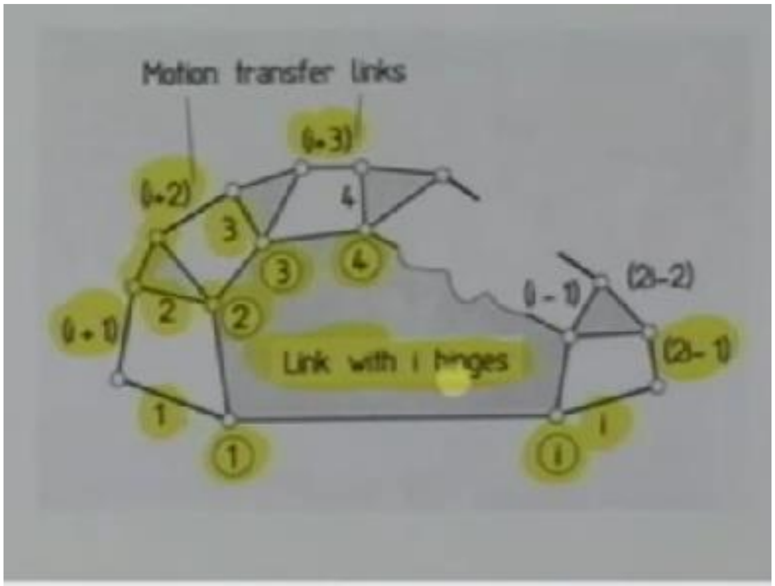

In this figure, we start with a link with i hinges. This is the link which has i hinges numbered as 1, 2, 3, 4 so on up to i. At each of these hinges we connect another link. Like at the 1st hinge we connect link 1, at the second hinge we connect link 2, at the third hinge we connect link 3 and so on this ith link at the hinge i. To connect these two links 1 and 2, we must have some motion transfer links, accordingly this, (i + 1), (i + 2) and (i + 3) so on up to (2i – 1). The thing to note that, at the hinge number 2, 3, 4 up to (i – 1), we have ternary links. Because link 2 has three hinges and that is true for all other links connected at hinge 3, hinge 4 and so on. Because, if we have binary link at 2 then this particular hinge will not connect to form motion transfer, because three links namely (i + 2) and another link number 2 will get connected at this higher order hinge. Then hinge as a simple hinge all these links starting from number 2, 3, 4 and so on up to (i – 1) must be ternary link. So we have produced a close chain minimum number of links if we start from a link with of ith order that is with i hinge.

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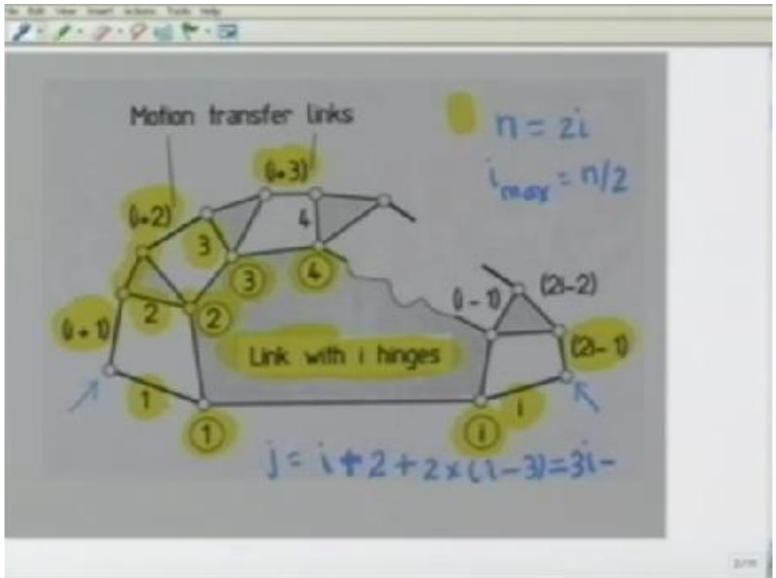

Now let me count total number of links n if we have started with, I have already shown the number up to (2i – 1) and we count this starting link which is having i hinges so the total number links is 2i. Thus, we see that if this is an ith order hinge then minimum I need 2i number of links to produce a closed chain. That means, total number of links is n then i_{max} can go up to n/2 and not more than n/2. I emphasize that is the possible value of i_{max}, not necessarily i_{max} has to be n/2, definitely it cannot be more than n/2.

Next thing we have to prove, that this closed chain, from this closed chain if I hold one link fixed it must produce a single degree freedom mechanism that is, this particular closed chain must satisfy our old Grubler's criterion. For that we count n = 2i. Let me count the maximum hinges j, we have started with i hinges on this initial link so j equal to i plus there is one hinge here and there is another hinge here which is at two plus on all other links two, three, four there are two external because these are all ordinary links one of the hinge has been already count with this starting link. There are two hinges extra hinges on each of it so that into i, how many such linkages? How many such links? We have starting from 2 to (i – 1) that is 2(i – 3). So, the number of hinges

$$j = i + 2 + 2(i – 3) = (3i – 4).$$

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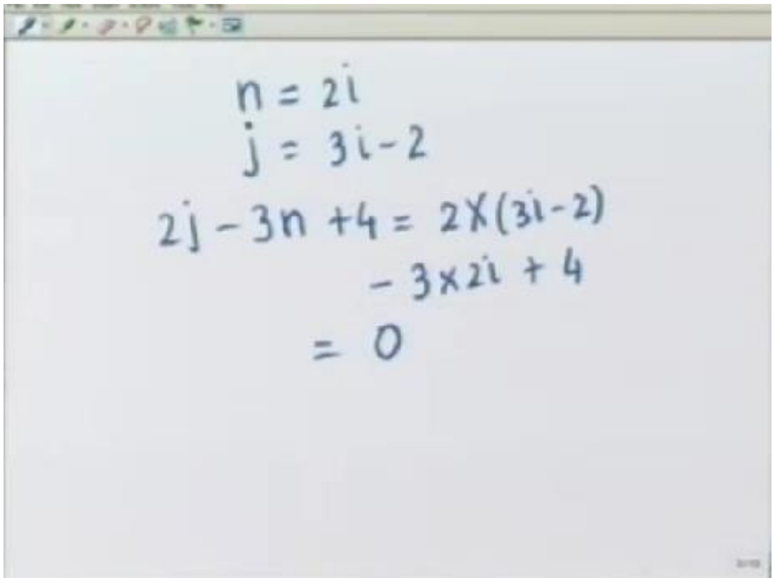

So we see that in this closed chain, the total number of links n turns out to be 2i, where i denote the highest order link in this chain. The total number of joints j = 3i – 2. If we write the Grubler's criterion that is,

$$2j - 3n + 4 = 2(3i - 2) - 3*2i + 4 = 0.$$

Thus the Grubler's criterion is satisfied by this closed finite chain and consequently this can constitute a single degree of freedom planar linkage. So we concentrate on these two results that we have just now derived.

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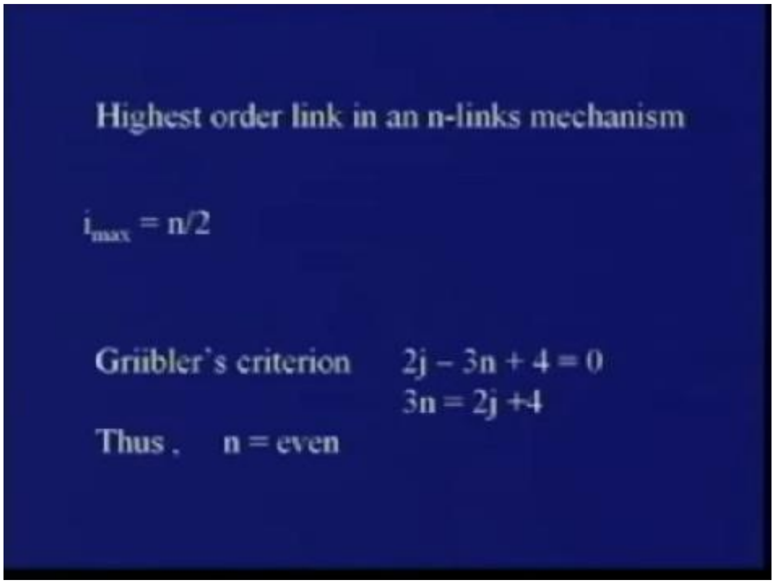

One is that the minimum number of binary links in a linkage must be four and the second is that is highest order link n^{th} link mechanism that is i_{max} is n/2. Since, all the single degree freedom linkage must satisfy the Grubler's criterion that,

$$2j - 3n + 4 = 0 \Rightarrow 3n = 2j + 4$$

Now we note that the right-hand side 2j + 4 is an even number and if 3n is equal to an even number then the n must be even, which means all the planar linkages simple pairs and single degree of freedom must have even number of links. We have already seen that the four-link mechanism is the simplest mechanism. So the next more complicated mechanism should be n equal to 6 that is a six-link mechanism. If the kinematic requirements are little more complex, which cannot be satisfied by a four-link mechanism then we have to try to use a six-link mechanism. Now let me go into this number synthesis of six-link mechanism.

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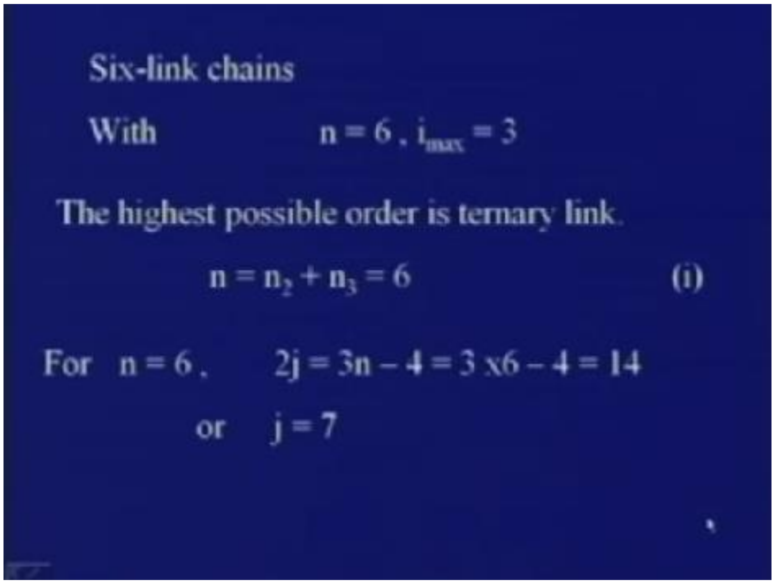

With a six-link chain we have n = 6 that is, $i_{max} = n/2 = 3$. So the highest possible order is a ternary link. So a six-link mechanism constitutes a binary links and ternary links. So the total number of link n equal to n_2 (that is the number of binary links) plus n_3 (that is the number of ternary links) is equal to 6. For n = 6, we know to satisfy Grubler's criterion, 2j = 3n – 4 that is 3∗6 – 4 = 14, that is j = 7.

We have got one equation, numbered equation one number in terms of two unknown in terms of n_2 and n_3. So we derived another equation involving n_2 and n_3 by counting the number of elements.

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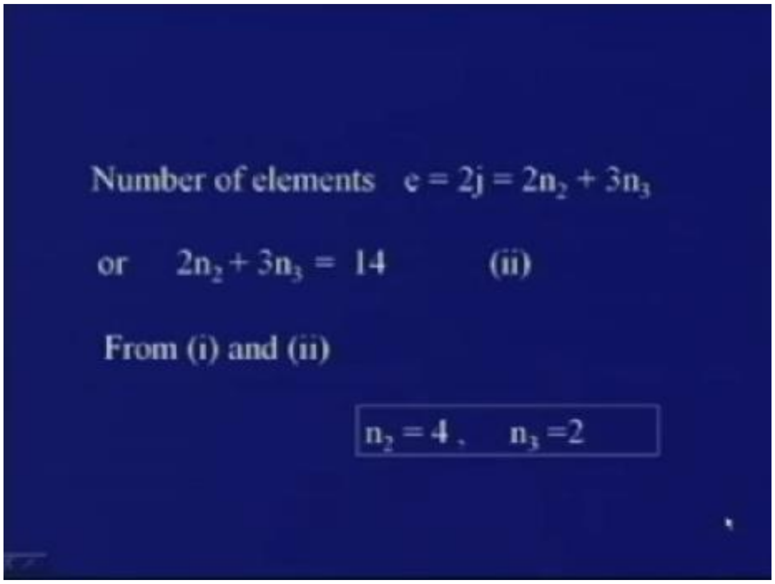

The number of elements e given by 2j which is also given by $2n_2 + 3n_3$. Thus, $2n_2 + 3n_3 = 2j$, where j = 7 this turns out to be 14. This is the second equation involving these two unknowns namely n_2 and n_3. Our previous equation was $n_2 + n_3 = 6$ and the second equation is $2n_2 + 3n_3 = 14$.

We can easily solve these two linear equations in two unknowns namely n_2 and n_3 as $n_2 = 4$ and $n_3 = 2$. Thus, a six-link mechanism has four binary links and 2 ternary links. Now we shall see what are the possible combinations of these binary and ternary links to generate six links, different types of six-link mechanisms?

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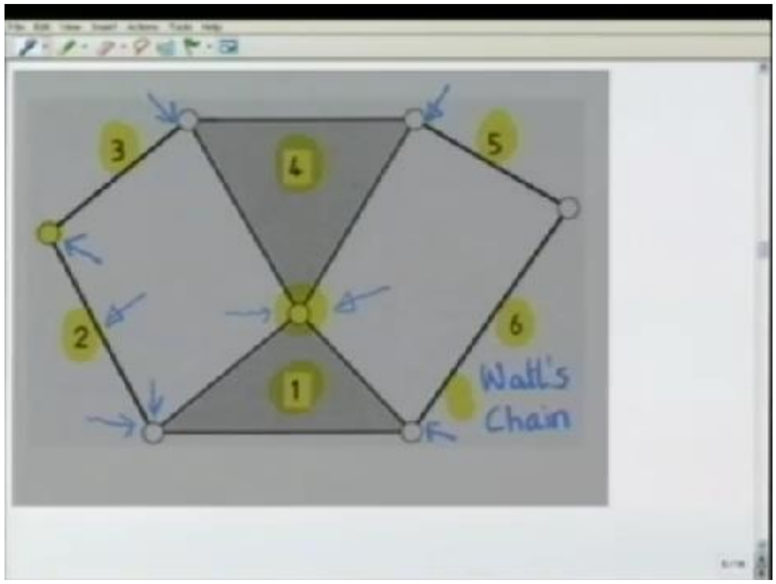

Above figure shows one possible six-link chain with two ternary links and four binary links. As we see, the link number 1 is a ternary link, link number 4 is another ternary link whereas link number 2, 3, 5 and 6 are all binary links. The thing to note that in this chain there are six-link and seven revolute pairs, six revolute pairs we can count at vertices of this hexagon and one inside the hexagon. Another thing to note that here the two ternary links 1 and 4 are directly connected by revolute pair and all the four binary links are connected to the ternary links. This chain is known as Watt's chain.

So in a Watt's chain, two ternary links are directly connected to each other. In this Watt's-chain, we can see that the two ternary links that is number 1 and 4 are equivalent. In the sense, both of them are connected to a ternary link at one kinematic pair and two two binary pair at the other two revolute pairs. Like 4 is connected to link number 4 by a revolute pair, to the binary link 4 is connected to another binary link 3 at revolute pair and is connected to the ternary link 1 at revolute pair. Exactly same thing happened for the link number 1, it connected to the ternary link 6 to this revolute pair and binary link 2 to this revolute pair and to ternary link 4 by this revolute pair.

Thus topologically there is no difference between link number 1 and 4. The same is true for all binary links namely 2, 3, 5 and 6 each one of which is connected to a ternary link at one end and to a binary link at the other end. For example, link number 2 is connected to a ternary link at one end and to a binary link at the other end and the same is true for all other binary links.

Thus, there are two types of links ternary links and binary links but both the ternary links are equivalent and all the four binary links are also equivalent. So from a Watt's-chain by kinematic inversion that is depending on which link we hold fixed we can get two different types of Watt's mechanism. One type of Watt's mechanism we can get by holding binary links fixed with 1, 2, 3 or 5 and 6, because all of them are equivalent and the second type of Watt's mechanism we can get by holding one of the binary links that is either one or four are fixed.

We will now show a model of a six-link Watt's mechanism where we will find, that one of the binary links is held fixed.

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As an example of Watt’s mechanism with a binary link fixed let us go back to our old example of this parallel jaw plier. We hold this lower jaw that is this blue link fixed, this is a binary link because it has two revolute pairs. Now let us note that this binary link is connected to another binary link at this revolute pair and to this ternary link at this revolute pair. This lower jaw is a ternary link because it has three revolute pairs and this small link is another ternary link which has three revolute pairs and these two ternary links are directly connected. So it is one type of Watt’s chain where we know two ternary links must be directly connected. If we hold this lower jaw fixed then we are holding this binary link fixed. We should also know that this is binary link, this upper jaw is a binary link and this below link is another binary link. This binary link is connecting this ternary link and this binary link. As a result of this we get a Watt’s mechanism by Watt’s-chain. This is Watt’s mechanism of one kind. Later on we will see Watt’s mechanism of another link where ternary link will be held fixed.

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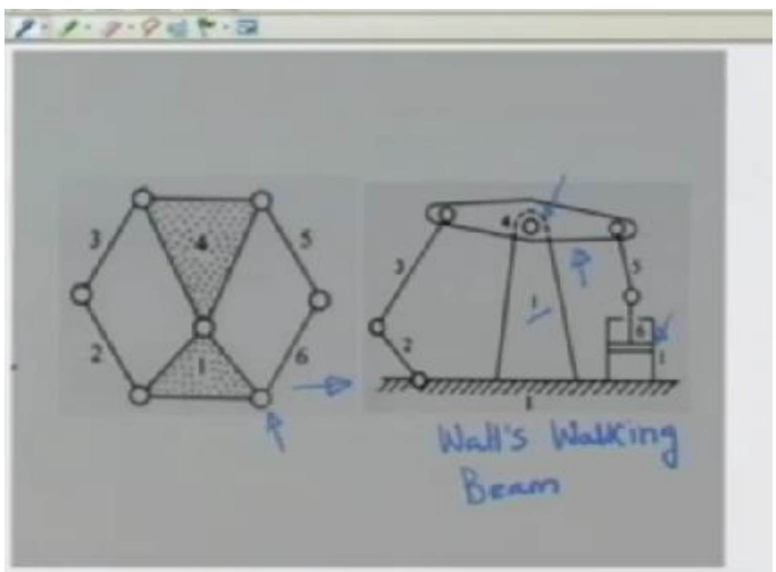

An example of another type of possible Watt’s mechanism let us consider above figure. Here as we started from a Watt’s chain which has two ternary links 1 and 4 and 4 binary links 2, 3, 5 and 6. Here, is a binary link 1 which is held fixed and this is known as Watt's walking beam engine. In this Watt's walking beam engine, we must see that in the chain we have shown revolute pair between 1 and 6 which has been replaced by a prismatic pair between the cylinder and the piston. But in our analysis, we always treated revolute pair and prismatic pair as equivalent. So here, as we see that this great beam that is link 4 is connected to ternary link directly by this revolute pair. This is another type of Watt's mechanism which is possible to get by kinematic inversion from a Watt's chain.

An another example of a six-link mechanism with seven hinges we can get the following figure.

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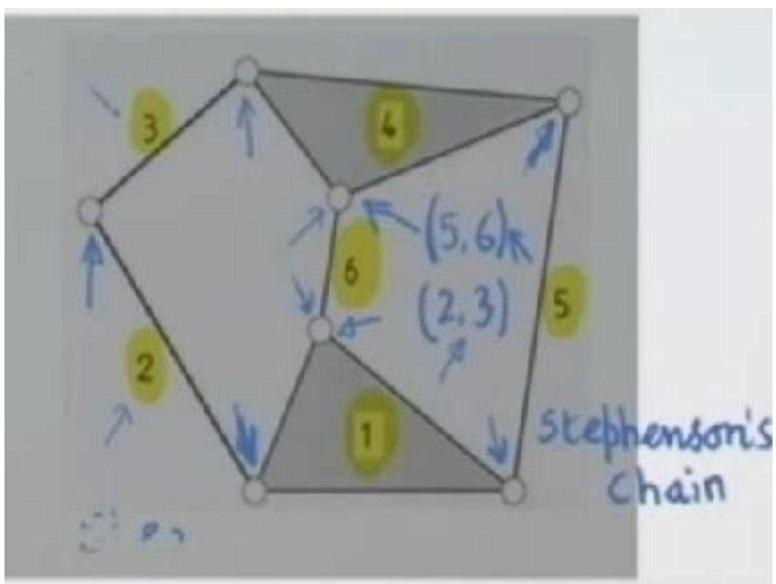

Here, as we see there are two binary links namely, two ternary links namely 1 and 4. However, unlike in a Watt's-chain these two ternary links are not directly connected to each other rather they are connected via binary link number 6. Here, we have two ternary links 1 and 4 which are connected by a binary link 6, binary link 5 and by two binary links namely 2 and 3 and this particular chain where the two ternary links are not directly connected is known as Stephenson's chain. So we see that in a Stephenson's chain two ternary links are not directly connected. In a Stephenson's chains the ternary links 1 and 4 are equivalent in a sense that both 1 and 4 are connected to three binary links and three revolute pairs. For example, link 1 is connected to binary link 6, binary link 5 and binary link 2 at these three revolute pairs and link number 4, the other ternary links is also connected to three binary links to link number 5 here, link number 6 here and number 3 here. Thus both these ternary links are topologically equivalent because both of them are connected to three binary links.

However, so far the binary links are concerned there are two varieties, namely 5 and 6 and 2 and 3. We should note that both 5 and 6 are connected to two ternary links at two joints, link 6 is also connected to two ternary links at two joints. However link number 2 and 3 at one end is connected to a ternary link but at the other end it is connected to a binary link. So there are two types of binary links they can be grouped as (5, 6) and (2, 3). So by kinematic inversion we can get three different types of Stephenson's mechanism depending on whether the ternary link 1 or 4 is held fixed or one of the binary links in this group that is either 5 or 6 is held fixed or one of the other group namely 2 and 3 that is either 2 or 3 are held fixed. So there are three different types of Stephenson's mechanism which can be obtained by kinematic inversion from the same Stephenson's chain. We now see a model of a Stephenson's chain to generate a Stephenson's linkage where a ternary link is held fixed.

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Let us now look at the model of this Stephenson's mechanism where one of the ternary link is held fixed. Here we have this fixed link as the ternary link which has three hinges one here, one there and another there and this is the other ternary link which is connected to the fixed link by two binary links. So the two ternary links are not directly connected, they are connected via binary links at these two points and by two binary links at this point, this is a binary link this is a binary link. These binary links are equivalent because at one end this binary link is connected to a ternary link, at this end this binary link is connected to another binary link. Similarly, this binary link is connected to a ternary link at this end and at to a binary link at this end. So these two binary links are of same nature. Similarly these two binary links are also of same nature because they are connected at both ends to ternary links. So one of the ternary links is held fixed and we get one variety of a Stephenson's linkage.

Another example of Stephenson's linkage let us consider the same Stephenson's chain and consider one of the binary links to be fixed.

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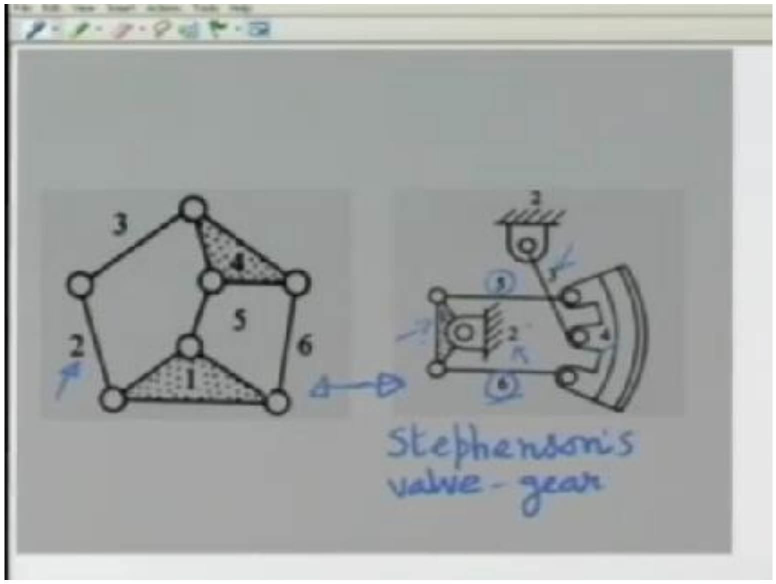

As shown in above figure, this is the Stephenson’s chain that we have considered earlier and in this chain if we hold a binary link say link number 2 fixed then we get this mechanism. As we see, 2 is connected to ternary link 1 and link 2 is the fixed link which is connected to a binary link 3 and to a ternary link 1. Link number 4 is the other ternary link which is connected to link 3 here, link 6 here and link 5 here. This is known as Stephenson’s valve gear mechanism which is used in a steam engine. We have seen, a six-link chains consists of four binary links and two ternary links and various combinations which are possible as Watt’s linkage or Stephenson’s linkage.

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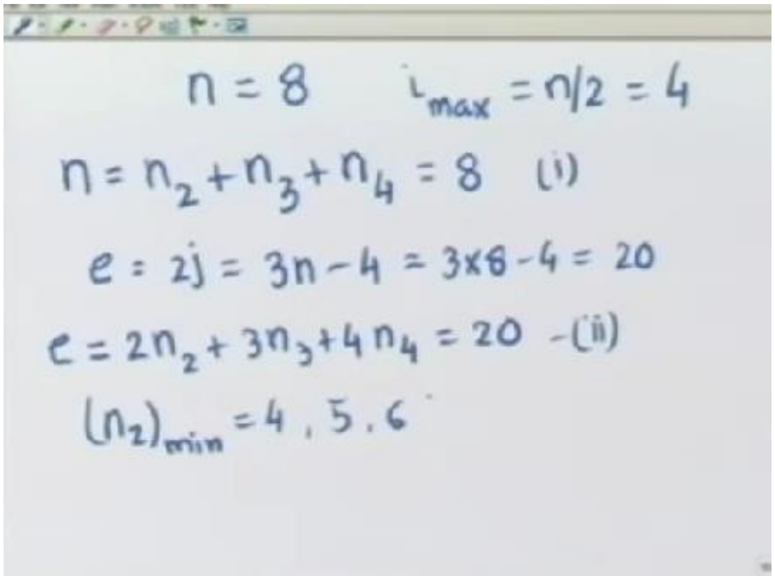

Let us consider the next higher order link possible to give you the flavour of number synthesis. Obviously the next most complicated mechanism we consider with ‘n’ even is an eight-link mechanism that is n = 8. Consequently, highest order link possible in an eight-link mechanism that is $i_{max} = n/2 = 4$. So an eight-link mechanism we will have binary link, it is possible to have ternary link and it is possible to have quaternary link.

If the number of binary links is n_2, the number of ternary links is n_3 and the number of quaternary links is n_4, then the total number of links $n = n_2 + n_3 + n_4 = 8$. This is our first equation to determine n_2, n_3 and n_4. We also know that the number of elements, $e = 2j = 3n - 4$, so that Grubler’s criterion is satisfied, which means $3*8 - 4 = 20$. Counting the number of elements from the viewpoint of links, we can write $2n_2 + 3n_3 + 4n_4 = 20$, because this is also equal to the number of elements. So, this is our second equation 2.

It may now appear that we have three unknowns, n_2, n_3 and n_4 to determine, but we have only two equations namely 1 and 2. Thus there may be infinite solutions. A little thought would convince us that is not the situation we still have finite number of solutions because we should remember all these numbers n_2, n_3 and n_4 are integers not only that the minimum value of n_2 is also 4. Number n_2 can start from 4 then can go up to 5, 6 and so on.

Let us see what are the various solutions possible to these two equations that under such restrictions that all these numbers n_2, n_3 and n_4 must be positive integers there is no point having a negative number for the number of links and also that minimum values of n_2 is 4.

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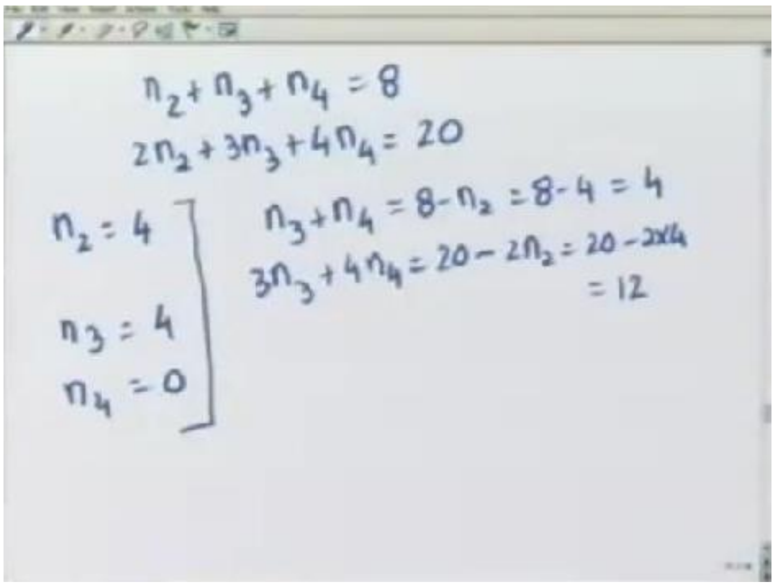

So for an eight-link mechanism, we have got two equations namely,

$$n_2 + n_3 + n_4 = 8 \text{ and } 2n_2 + 3n_3 + 4n_4 = 20.$$

If we assume that the value of $n_2 = 4$ then from these two equations we get, $n_3 + n_4 = 4$ and from the second equation we get $3n_3 + 4n_4 = 12$. Now we get these two equations to solve for n_3 and n_4 and the obvious solution is $n_3 = 4$ and $n_4 = 0$. That means we can get an eight-link mechanism consisting of 4 binary links and 4 ternary links. There is no necessity that we must have a quaternary link.

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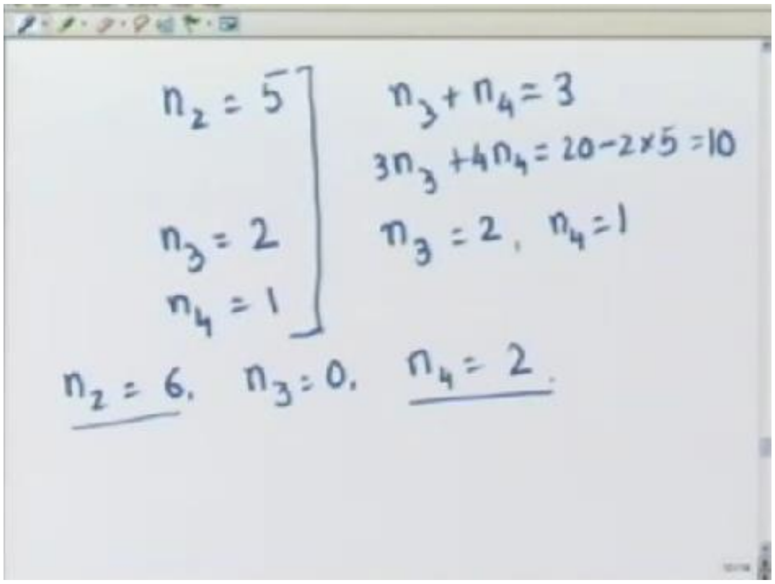

However if we consider $n_2 = 5$ then we get $n_3 + n_4 = 3$ and $3n_3 + 4n_4 = 10$. Solving these two equations, we get $n_3 = 2$ and $n_4 = 1$. Thus, we can also have an eight-link mechanism with 5 binary links with 2 ternary links and 1 quaternary links. Similarly, if we take $n_2 = 6$, one can easily find that we will get $n_3 = 0$ and $n_4 = 2$. That means, we can have an eight-link mechanism with 6 binary links and 2 quaternary links. From these three different types of eight link chains by kinematic inversions one can get a very large number of different mechanisms.

In conclusion, let me now repeat the foremost important points that we have learnt today during this discussion of number synthesis of planar linkages. The first point is that the minimum number of binary links in any such linkage must be four: that means, we must have at least four binary links. The second point is that the highest order link in an nth-link mechanism is n/2, that is in a six-link mechanism the highest order is ternary link and in an eight-link mechanism the highest order is quaternary link. Third thing we have seen that, the total number of links must be even and the last point is that with increase in the number of total links the possible types of various mechanisms that we can have from some such chains by kinematic inversion increases drastically.