Kinematics of Machines

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Module – 12 Lecture - 3

We finished our last lecture with the contact ratio for a pair of involute gears. Today, we start our discussion with the expression of the contact ratio for a rack and pinion drive.

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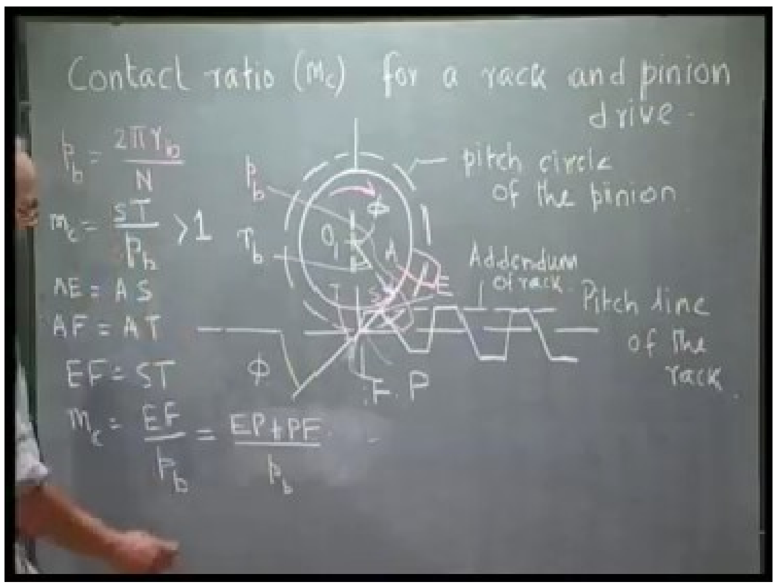

We use the symbol m_c for contact ratio of a rack and pinion. Let's say, this is the pitch circle of the pinion. As we know for the rack the pitch circle radius is infinity, that is the pitch circle gets converted into a straight line. As in the case of a pair of gears, two pitch circles were touching each other, hear again the pitch line of the rack acts as a tangent to the pitch circle of the pinion. So, this is pitch line of the rack and this is the pitch circle of the pinion. If we draw the base circle of the pinion, this point of contact between the pitch circle and the pitch line is the pitch point, we denote it by P. Let me draw a tangent to the base circle from this pitch point. This line represents the line of action. Let this point of tangency be denoted by A, this is the centre of the pinion, then this angle is the operating pressure angle φ which is same as this operating pressure angle. The point of contact between the teeth of the pinion and the rack always moves along this line of action.

Let me talk of a particular tooth on this pinion, which is just beginning the engagement, that means the contact with that particular tooth is just starting at this configuration. Let this be the tooth which is coming in contact with the tooth of the pinion at this instant. Where does the contact start? Contact obviously starts at the addendum line of the rack that is the highest point of the rack tooth. So if I draw addendum, this is the addendum line of the rack. The rack tooth is perpendicular to the line of action. The contact starts at this point between the pinion tooth and the rack tooth on the line of contact; this is the addendum line of the rack. If I complete the rack tooth, it will look something like this. This is the rack. At the beginning of the contact, this point let me call E. This pinion rotates in the clockwise direction and drives the rack to the left.

Now when does the contact between this pair of teeth is lost? It is lost when the contact comes at the addendum circle of this pinion. Let this be the addendum circle of this pinion tooth. When this addendum circle intersects this line of action, let's say this point we call F. This tooth has rotated and let's say this is the addendum circle. Now how much is the movement of this tooth measured along the base circle, let me call this point S. The tooth has rotated and the movement along the base circle is from S to T. This is due to the rotation of the gear; this point S of this tooth has moved up to this point T.

Tto ensure that the teeth remain in contact the next tooth that is the adjacent tooth on this gear, this point must get into beyond this point. This point must move beyond S such that this tooth comes in contact with the next tooth and this distance measured from this point to this point. Let me mark it, the distance between these two points on adjacent tooth along the base circle, that is what we denoted as base pitch or p_b.

The distance between two adjacent teeth along the base circle that means from this point to this point, that is what we call p_b. An expression for p_b we have used last time, which is $2\pi r_b/N$; where r_b is this base circle radius of the pinion, OA is r_b, and N is the number of teeth on the pinion. Contact ratio m_c was defined as: the movement of one tooth along the base circle, which is ST divided by the base pitch. As we said, this must be more than one so that continuous transmission of motion between these two rack and opinion is ensured. To get the expression of this m_c, the contact started at E and finished at F and these are involute profiles. So, AF is nothing but AE, AE is the string length which was wound on to this base circle and by unwinding we have generated this involute profile. Because it is an involute profile, the distance AE is same as the distance measured along the arc of the base circle AS. So, AE = AS.

Similarly, this is also an involute from the same base circle, this is the same tooth. So AT measured along the base circle will be AF, that is the length of the string which has been unwound from this base circle. So AT = AF, length of the string AF is the length of the circular arc from A to T. So if we subtract, AF – AE = EF and if we subtract AS from AT, we get ST, so ST is same as EF. So contact ratio is nothing but EF/p_b. EF, we can write as, EP + PF. EP is this distance, PF is this distance. So m_c = (EP + PF)/p_b. Now we add PF, AE and EP and subtract AP to the numerator. So, we get,

$$m_c = \frac{EP + (PF + AE + EP) - AP}{p_b} \quad .$$

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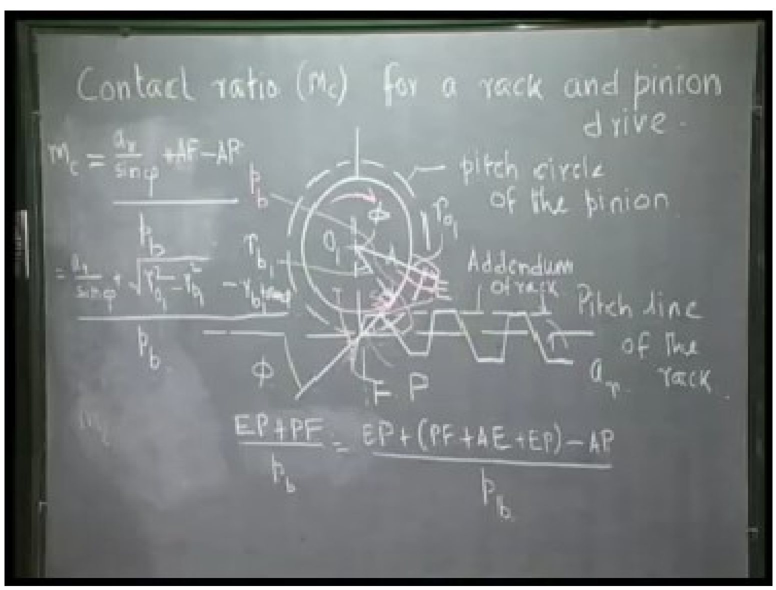

Finally, we get the expression of contact ratio $m_c = \frac{EP + (PF + AE + EP) - AP}{p_b}$. Here, what is EP? EP is this height, which we can see, if this angle is φ and this vertical height is addendum of the rack, which we denote by a_r. So this EP =, where φ is the operating pressure angle. PF + EP + AE is nothing but AF and $m_c = \frac{\frac{a_r}{\sin\varphi} + AF - AP}{p_b}$. A is this contact point between the tangent from the pitch point to the base circle, so this angle is 90^0. This is radius, this is tangent, so this angle is 90^0 and O_1P is nothing but the outer radius of the pinion. This is addendum circle, so O_1 to F is nothing but the outer radius of the pinion. So $AF = \sqrt{r_{O_1}^2 + r_{b_1}^2}$ where $r_{O_1}^2$ is the outer radius or addendum circle radius of the pinion and $r_{b_1}^2$ is base circle radius. This is φ and this is r_{b_1}, tan φ = AP/ r_{b_1}, so AP = r_{b_1} tan φ.

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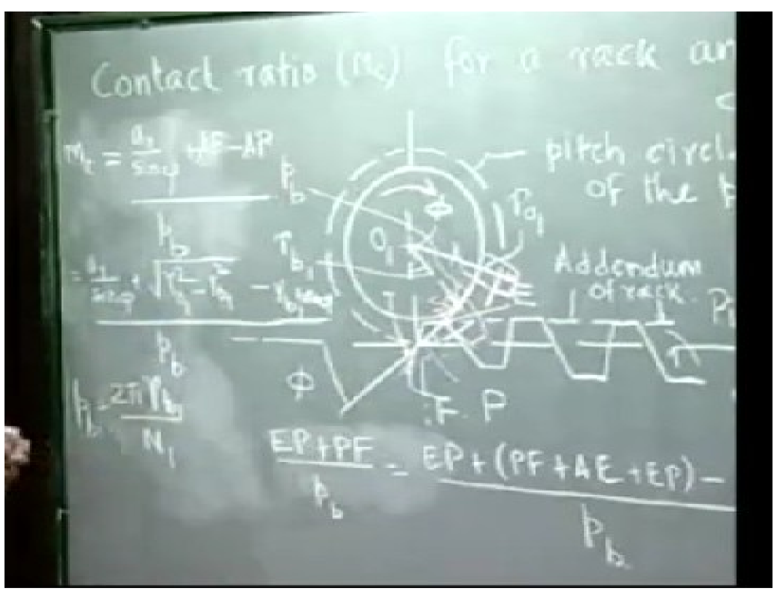

So we get, $m_c = \frac{\frac{a_r}{\sin\varphi} + \sqrt{r_{O_1}^2 + r_{b_1}^2} - r_{b_1}\tan\varphi}{p_b}$. Finally, we have got the expression for the contact ratio for a rack and pinion drive involving the addendum of the rack, operating pressure angle, outer radius of the pinion, base circle radius of the pinion and the base pitch(p_b), which is nothing but $2\pi\ r_{b_1}/N_1$, where r_{b_1} is the base circle radius and N_1 is the number of teeth of the pinion. This completes our discussion on the contact ratio for a rack and pinion drive. The methodology is just similar to what we used for a pair of involute gear teeth profile.

Now that we have finished discussion on contact ratio, let us start with a very important parameter for gear tooth geometry, that is known as interference and undercutting.

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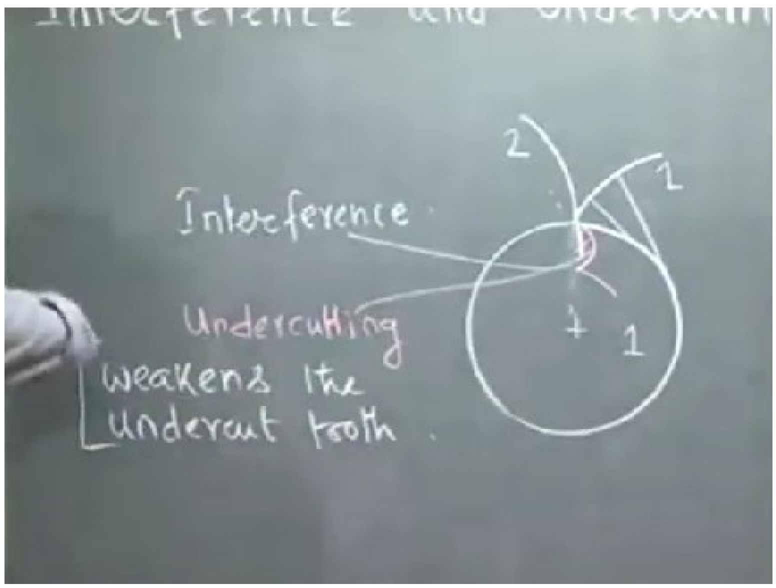

First, let me explain what we mean by interference and undercutting? If we remember from the very definition of involute profile, it is obvious that the involute does not exist inside the base circle. It is from unwinding from the base circle outward, we created the involute profile. Let's say, it starts from this point A radially and as the string unwinds, we generate the involute profile. So the involute profile exists only outside the base circle, does not exist inside the base circle. Suppose, we use very few numbers of teeth on a gear and the teeth size is very big, then it may so happen that the addendum circle of the mating gear gets into the base circle of the other gear. If the mating profile, which is another involute for another base circle gets inside the base circle.

As we see, there is no involute of this gear, say this is gear 1 and this is 2. The gear 1 does not have any involute inside this base circle. Whereas, the involute of the other mating gear is getting into the base circle so there is no conjugate profile. So one can extend this involute of gear 1 may be radially, even then there is interference. As we see, this tooth will interfere with this tooth because this tooth at once to occupy this position, whereas this tooth is lying there, this is called interference. One solution to this interference is to remove some portion of gear 1 – we do not have the involute profile of gear 1 anyway – so let me cut this portion from gear 1. Then this gear tooth is not interfering with gear 1, the tooth of gear 2 is having a space inside gear 1, though there is no conjugate profile to maintain contact. However, the conjugate action may be taken care of by the other pair of teeth in engagement, so the continuous motion is transmitted maintaining the constant angular velocity ratio, but this tooth is not transmitting any power because this is getting into this portion and this is what is called undercutting. We have avoided the interference by undercutting the tooth of gear 1, inside the base circle. But this is not a very satisfactory solution, because undercutting weakens the undercut tooth and it is here at the root of the tooth where failure takes place so if we have weakened the tooth near its root, it is not a very desirable solution, only if it is a must then only we should undercut the tooth.

Next, we shall discuss how to ensure that there is no interference and how to avoid interference by ensuring some minimum number of teeth on a gear. We cannot have number of teeth on a gear less than a minimum number to avoid interference – that will be our next topic of discussion. As we have just now shown the possibility of interference between a pair of gear teeth because involute does not exist inside the base circle and not so satisfactory solution was to undercut the tooth.

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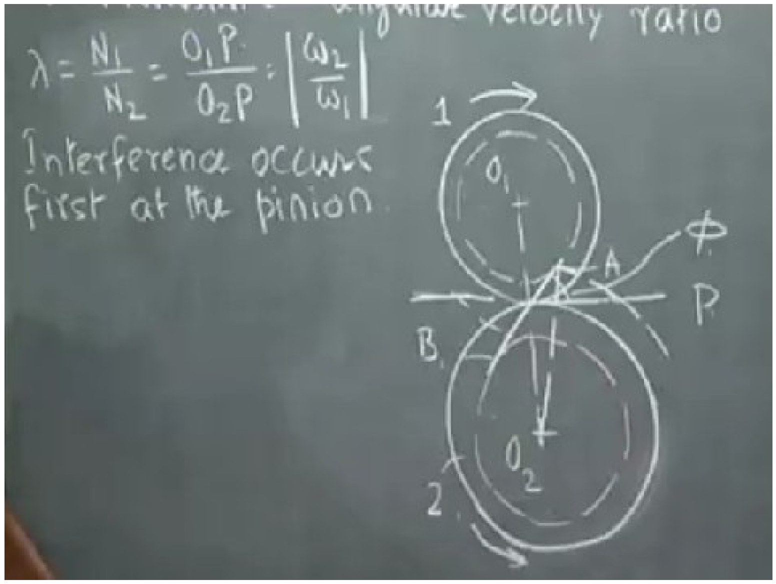

A better solution is to have a minimum number of teeth on a gear to avoid interference. We have already seen how the interference takes place because the addendum circle gets into the base circle. So now, we talk of a minimum number of teeth to avoid interference to transmit a particular angular velocity ratio – say we call it λ, which is defined as ω_2/ω_1 which is less than equal to 1, that means ω_1 is more than ω_2. When we transmit continuous rotation from one shaft to another at the constant angular velocity ratio, let us forget about the sign for the time being, I defined angular velocity ratio only by the positive number and I define it in such a way that it is less than equal to one, that means the higher speed is at the denominator. For example, suppose λ is 8/9. To transmit a constant angular velocity 8/9, I can have 16 and 18 teeth or 24 and 27 teeth or 32 and 36 teeth. The question is, what must be the minimum number of teeth to transmit angular velocity 8/9? Can I use 8 and 9 teeth to transmit constant angular velocity 8/9. The question we are trying to answer – what is the minimum number of teeth such that interference is avoided while transmitting a particular angular velocity ratio which is given to us, the value of λ? Towards this end, let me first draw the two pitch circles. Suppose these are the two pitch circles, ω_1 is the higher speed, so this is gear 1 which rotates at a faster speed and this is gear 2 which is rotating at a slower speed. Suppose, this gear is rotating this way and this gear is rotating this way, these are the pitch circles.

Now let me draw the two base circles. This point is O_1, this point is O_2, this is the pitch point which we call P and if we draw the common tangent to these two base circles, that is the line of action AB. Now as this is very clear, because this is a smaller gear AP is smaller than BP, this is a larger gear which is rotating at a slower speed. So $\lambda = N_1/N_2$, where N_1 is the number of teeth on this gear and N_2 is the number of teeth on this gear, which is also same as O_1P/O_2P, which is same as ω_2/ω_1.

Now the addendum circle of either gear should not get into the base circle of the other gear and as we know of the contact point is restricted to move on this line of action that is line AB, so which addendum circle will first get into the base circle. Addendum circles are of same size – both the gears teeth are of same size, they have the same addendum. So it is obvious, that the bigger gear addendum will first get into the base circle. If this is the addendum circle, if this touches A the addendum circle of this gear, which has exactly same addendum will pass through the below the point B, it will not get into the base circle. The point of contact always lies on this line, so I am only interested in seeing whether the addendum circle of this line intersecting AB beyond A or beyond B. So, interference obviously occurs first with the pinion when the addendum circle of the gear just passes through the point A. Because, even at that time the addendum circle of the pinion does not pass through B, it is still to interfere with this gear.

First point to note is that interference occurs first at the pinion, that is the smaller gear and we are considering as if the interferences has just started. From this diagram, we should be able to find out, what is the minimum number of teeth that is N_1 to avoid interference? And this common tangent to the pitch circles, this is the point P, and this angle is the operating pressure angle φ. Now this distance O_2A which is nothing but the radius of the addendum circle of gear 2.

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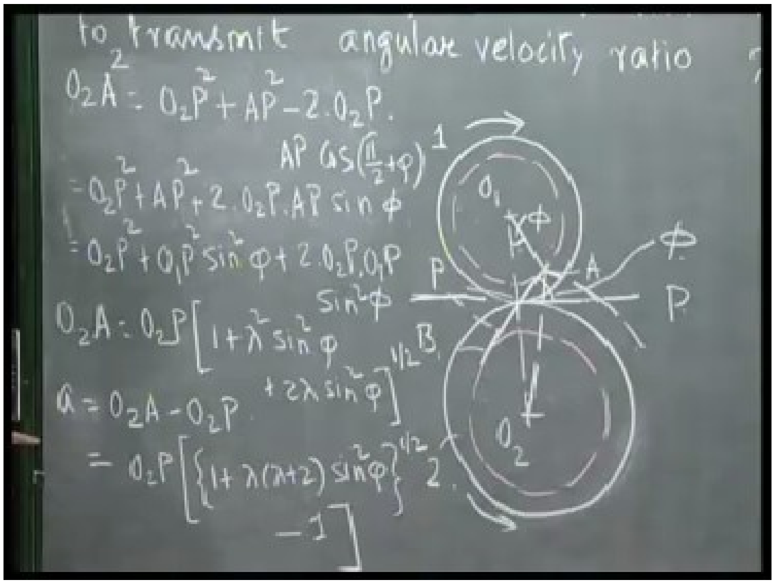

Let me first write an expression for this O_2A. We consider the triangle O_2PA and this point is P. So $O_2A^2 = O_2P^2 + AP^2 - 2\ O_2P$ ¿ AP ¿ $\cos(\pi/2 + \varphi)$ – using the triangle law, where $(\pi/2 + \varphi)$ is the angle between O_2P and AP. So, this we write $O_2P^2 + AP^2 - 2$ O_2P ¿ AP ¿ $\sin\varphi$ and what is AP? If we remember this angle is also φ. So, AP = $O_1P \sin\varphi$. So, this we write $O_2A^2 = O_2P^2 + O_1P^2 \sin^2\varphi + 2\ O_2P\ O_1P \sin^2\varphi$.

Taking O_2P common from this expression and take the square root, we get $O_2A = O_2P\ [1 + \lambda^2 \sin^2\varphi + 2\lambda \sin^2\varphi]^{1/2}$ where, $\lambda = O_1P/O_2P$. The speed ratio is nothing but the pitch circle radius ratio. So λ is that. $O_2A - O_2P$, O_2A is the addendum radius O_2P is the pitch circle radius, so the difference of this is nothing but addendum, $a = O_2A - O_2P$. If I use this expression of O_2A, we get this as,

$$a = O_2A - O_2P = O_2P\ [\{1 + \lambda^2 \sin^2\varphi + 2\lambda \sin^2\varphi\}^{1/2} - 1].$$

This is the expression for the addendum.

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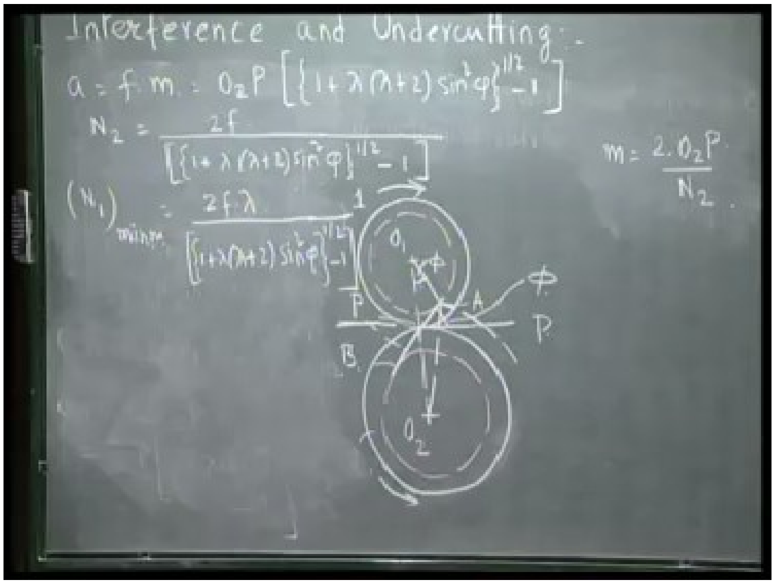

Addendum is always expressed as some fraction f of the module. If we remember, for the British Standard for 20^0 pressure angle, we said f is normally one, but we keep it as f, which is of the order of one. So ‘a’ = f ¿ m = O_2P [{$1 + \lambda^2 \sin^2 \varphi + 2 \lambda \sin^2 \varphi$}$^{1/2}$ – 1] and if we remember module was 2 r_P divided by number of tooth. Module by definition is pitch circle diameter by the number of teeth ie., number of teeth of the second gear, pitch circle diameter of the second gear. So this O_2P, we can write as $mN_2/2$. If we substitute O_2P in above expression, m cancels, so we get $N_2 = 2f/[\{1 + \lambda^2 \sin^2 \varphi + 2 \lambda \sin^2 \varphi\}^{1/2} – 1]$.

We have considered the situation where interference just started with the pinion and we will get this expression for N_2 which means if we multiply by N_1 by N_2, which is nothing but λ. N_1 minimum ($(N_1)_{min}$) tooth on the pinion, we get,

$$(N_1)_{min} = \frac{2f\lambda}{\left[\{1+\lambda(\lambda+2)\sin^2\varphi\}^{1/2} - 1\right]}$$

This is the minimum number of teeth required to avoid interference while transmitting an angular velocity ratio λ and λ we have defined as always less than 1. So, this is how we get the minimum number of teeth to just avoid interference. N_1 at least should be this much, if N_1 is more, then teeth will be smaller, so there will be no problem. This is just when the interference starts, so N_1 should be more than this $(N_1)_{min}$.

Just now, we derived the expression for the minimum number of teeth that must be on a pinion so that interference is avoided while transmitting constant angular velocity ratio λ.

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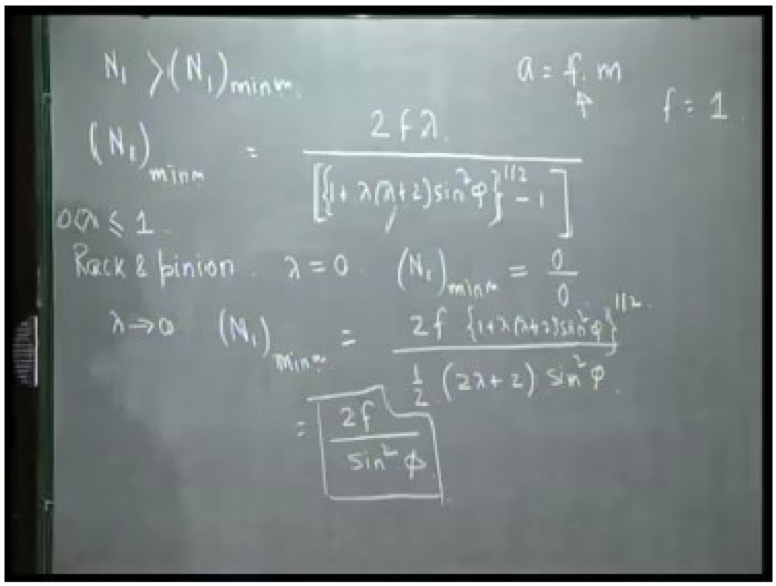

Number of teeth on a pinion must be greater than $(N_1)_{min}$, where $(N_1)_{min}$ was given by; where λ by definition is between 0 and 1. We always define λ as less than 1, that is the slower angular velocity divided by the faster angular velocity and φ is the operating pressure angle. What is f? f is addendum and expressed as a constant fraction of the module; f is normally 1 unless otherwise stated. From here let me try to drive the expression for the minimum number of teeth that must be there on the pinion while it is in contact with a rack and pinion. If it is a rack and pinion, then what must be the minimum number of teeth on the pinion to avoid interference. For rack and pinion, as we see λ is 0 because the angular velocity of the rack is 0, rack has only translational velocity. So λ is 0, ω_2/ω_1 is 0.

Now if we put λ = 0 here, then we get numerator and denominator 0, so this is 0/0 form. From this expression $(N_1)_{min}$ is standing out to be of the form 0/0. That is now nothing problematic as we know, we can always take the limit. Let us take the limit of this as λ tends to 0. So for rack and pinion, I can get $(N_1)_{min}$, by taking the limit of this expression as λ tends to 0. Because it is of the 0/0 form, we can use L'Hospitals rule, that means differentiate the numerator with respect to λ, denominator with respect to λ, then put λ equal to 0 and see what the limiting value is. So that way, if we differentiate the numerator with respect to λ, we get 2 f and if we differentiate the denominator with respect to λ, we get $\frac{\{1+\lambda(\lambda+2)\sin^2\varphi\}^{1/2}}{\frac{1}{2}(2\lambda+2)\sin^2\varphi}$. So as λ → 0 we get,

$$(N_1)_{min}=\frac{2f\{1+\lambda(\lambda+2)\sin^2\varphi\}^{1/2}}{\frac{1}{2}(2\lambda+2)\sin^2\varphi}$$

Now we put λ = 0, then we get, $(N_1)_{min}=\frac{2f}{\sin^2\varphi}$. So, this is the expression for minimum number of teeth on the pinion which engages with a rack and without interference. This expression of course we could have got geometrically very simply by drawing the rack, just undergoing interference with the pinion. That is what we are going to do next.

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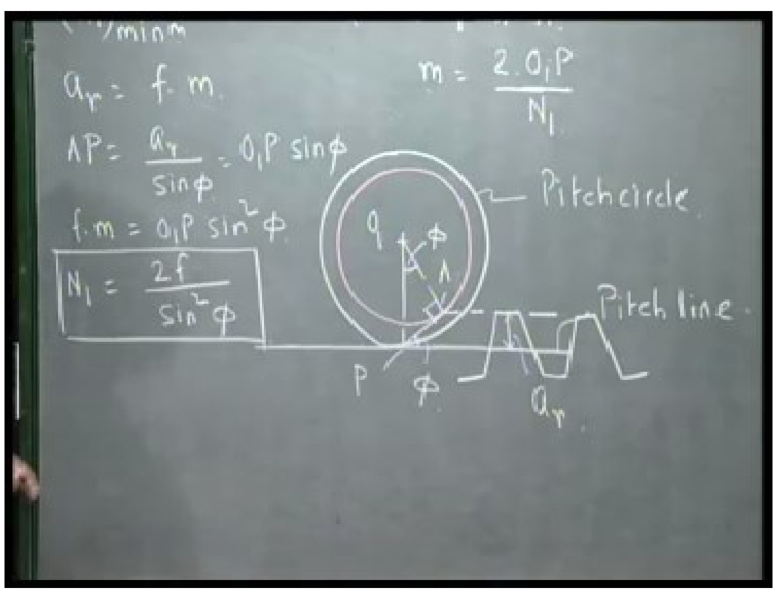

Just now we derived the minimum number of teeth on a pinion that is required to avoid interference with a rack, but that we did analytically from the expression.

Let me now get the same expression of the minimum number of teeth geometrically. So $(N_1)_{min}$ on the pinion for a rack and pinion to avoid interference. Towards this end, let me say this is the pitch circle of the pinion and this is the pitch line of the rack; this is the base circle of the pinion. If we draw a tangent to the base circle from this pitch point P, this angle is φ, this is O_1, this is A. Now if the addendum line passes through the point A, if this is the addendum line of the rack and that is the rack tooth looks something like this. If the addendum line intersects this base circle just at A, that is when the interference starts and we can use this diagram for getting the minimum number of teeth on the pinion. This is the rack of the addendum a_r which we write as f ¿ m and this angle is the operating pressure angle φ. So just when the interference starts, $AP = a_r/\sin φ$. And $AP = O_1P \sin φ$, this angle is 90^0. This is the base circle radius and this is the tangent, this angle is φ, pitch circle radius is O_1P and so $AP = O_1P \sin φ$. Therefore, a_r = f ¿ m = $O_1P \sin^2 φ$ and m is nothing but $2O_1P/N_1$, pitch circle diameter by the number of teeth, that is the definition of the module. So if we substitute m, we get $N_1 = 2f/\sin^2 φ$. This is the same expression, that we got earlier by taking the limiting value from the expression of $(N_1)_{min}$ for a pair of involute gears.

Next we shall solve an example to show the application of this expression that we got for $(N_1)_{min}$. How do we determine the minimum number of teeth that must be used to transmit a particular given angular velocity ratio λ?

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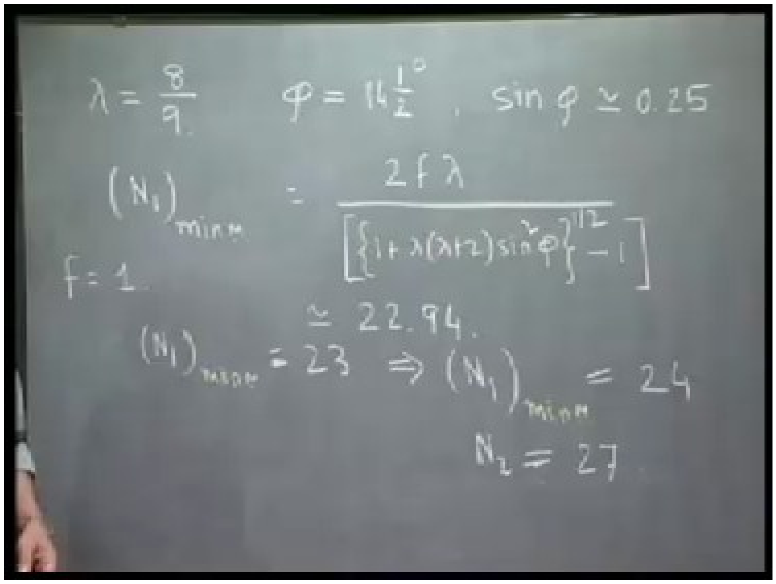

Let me now solve an example, for a pair of involute gears to transmit an angular velocity ratio λ = 8/9. The operating pressure angle φ, just as an example let me take it as 14½^0 which means sin φ ≈ 0.25. If we remember the expression for

$$(N_1)_{min} = \frac{2f\lambda}{\left[\left\{1+\lambda(\lambda+2)\sin^2\varphi\right\}^{1/2} - 1\right]}$$

. The most usual value of 'f' is. If we put f = 1, λ = 8/9 and sin φ = 0.25, from this expression $(N_1)_{min}$ turns out to be 22.94. But, number of teeth cannot be 22.94, it has to be an integer, so $(N_1)_{min}$ looks like it should be 23. But if N_1 is 23 and to transmit 8/9 angular velocity ratio, N_2 will turn out to be non-integer which means, this will imply N_1 has to be a multiple of 8, such that N_2 can be a multiple of 9 and they will be both integers. So $(N_1)_{min}$ = 24 and corresponding N_2 = 27. So, if we use 24 and 27 teeth, we can maintain a constant angular velocity ratio 8/9 and without interference. So, this is what we mean by minimum number of teeth to avoid interference.