Kinematics of Machines

Prof. A. K. Mallik

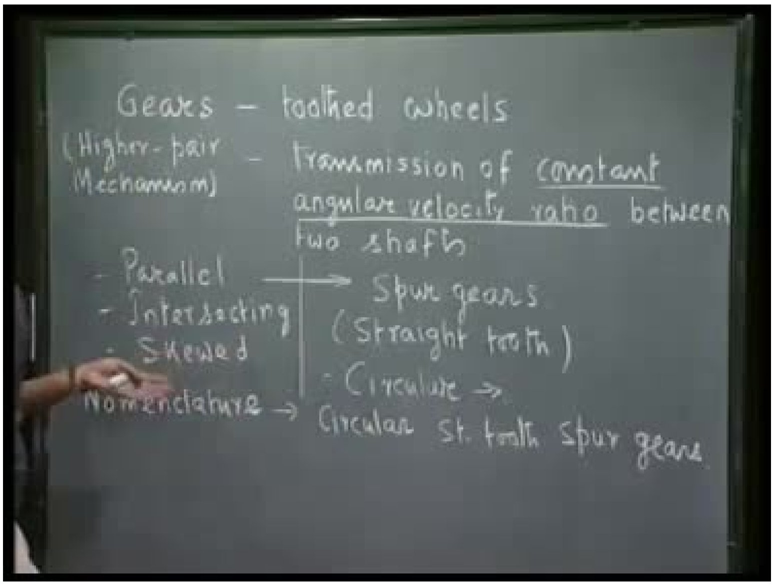
Department of Mechanical Engineering

Indian Institute of Technology, Kanpur

Module No. – 12

Lecture No. – 01

(Refer Slide Time: 00:33)



Today, we start our discussion on a new type of higher pair mechanism which are very commonly used and I am sure all of you must have seen this mechanism which are called gears. Gears are nothing but toothed wheels. These are used all over machines to transmit constant angular velocity ratio between two shafts.

Let me emphasize for transmission of constant angular velocity ratio between two shafts. Actually, a pair of gears are used, one gear is mounted on one shaft, let say the input shaft and the other gear mounted on the other shaft, that is the output shaft.

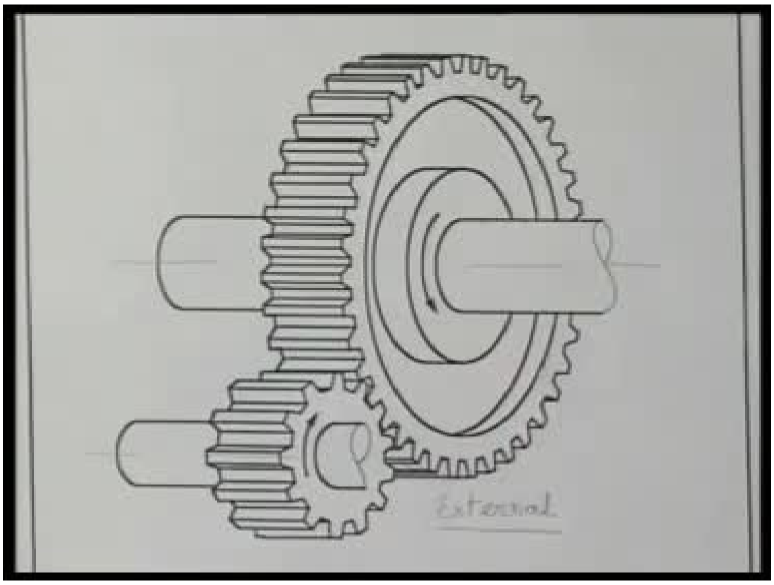
So, pair of gears are used to transmit a constant angular velocity ratio between two shafts. I want to emphasize this constant angular velocity ratio, it is very different from constant rpm ratio; rpm is not angular velocity, rpm is the average angular velocity or average angular speed, but gears are used, such that at every instant the instantaneous angular velocity of the two shafts maintain a constant ratio, thats why we use gears.

These two shafts can be parallel shafts or intersecting shafts, that means the shaft axis might be intersecting at a point or even skewed i.e., neither parallel nor intersecting shafts.

The spatial position of these two shafts can be either parallel or intersecting or skewed and they can be connected by a pair of gears to transmit constant angular velocity ratio. To start with we take a very simple type of gear to connect two parallel shafts which are called spur gears and that to with straight tooth, that means the tooth on these wheels are parallel to the axis of the shaft. These gear wheels are cylindrical and their teeth are cut on the surface of the cylinder and these teeth run parallel to the axis of the cylinder. For such spur gears, as we see the cylinders in a projected view, if we view it from the front along the axis of the shaft we see just circle. So, these are called circular gears, see instead of cylinder I will use the word circular.

Let me define all the terms which are used for this description of gears, gear geometry and gears transmission. So, we start with nomenclature of gears, by doing that I will define all the relevant terms with reference to circular straight tooth spur gears. So, now onwards, we may not emphasize that we are talking of all the time circular straight tooth spur gears, that will be assumed. Later on, we will talk about different types of gears, which are not circular straight tooth spur gears. Let me now show you with a figure, how these pair of toothed wheels which are circular straight tooth spur gears transmit motion from one shaft to another, maintaining a constant angular velocity ratio.

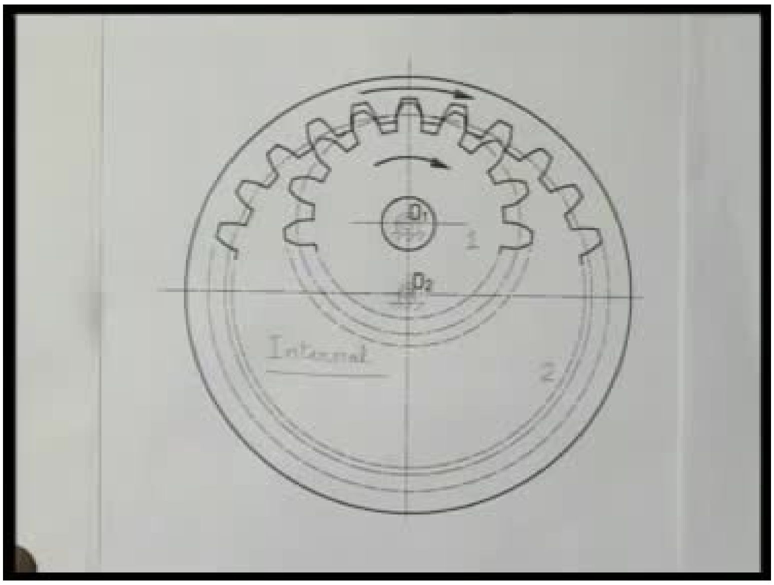
(Refer Slide Time: 05:16)



These figure shows a pair of spur gears. As we see, the teeth on these gears are parallel to the axis of these shafts; these are the teeth, these are the gear blanks, on the surface of this gear blank the teeth are cut or machined and this is one shaft on which this gear is mounted, this is another shaft where another gear is mounted. This kind of gear connection is called external gearing and as it rotates, this shafts along with this gear rotates in the clockwise direction, then the mating gear and the connected shaft will be rotate in the opposite direction.

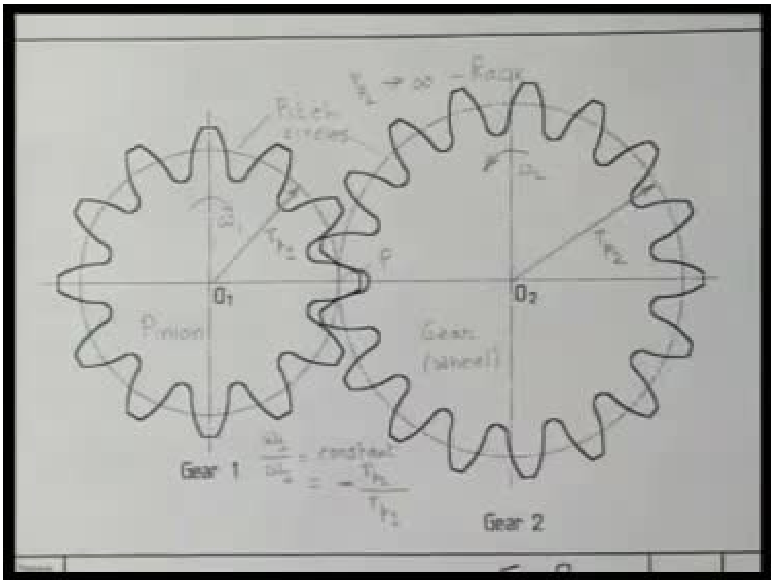
So the direction of the angular rotation is reversed due to this external gearing. Similar gears can also be used in internal fashion, such that one gear drives the other gear in the same direction maintaining the constant angular velocity ratio. This is the tooth of the gear on that the gear blank or body of the gear.

(Refer Slide Time: 06:40)



This figure shows a pair of internal gear; there are two shafts, there is a bearing between the shaft of the machine and the shaft pertaining to this gear 1, there is revolute pair here this is gear 2, which has a revolute pair with a fixed link 3; so, this constitutes three link mechanism. As we see, if this we call this gear 1 and this we call gear 2, the teeth are cut on external surface of gear 1 on this blank; the teeth are cut or machined on internal surface of gear 2 and these teeth of gear 1 gets into engagement with the teeth of gear 2. As it rotates, if gear 1 rotates in the clockwise direction, gear 2 also rotates in the same direction. So, this is what we call internal gearing.

(Refer Slide Time: 08:21)



So, we have seen external gears and a pair of internal circular gears, now let me define different terms of circular straight spur gears. This figure again shows a pair of external gearing connecting parallel shafts 1 at O_1 and the other at O_2. The smaller of these two gears, that is here, the smaller gear is called pinion and the larger gear either wheel or gear. Now let us observe these two circles, which are drawn on gear 1 and gear 2 touching each other and the centers of these gears are at O_1 and O_2 respectively.

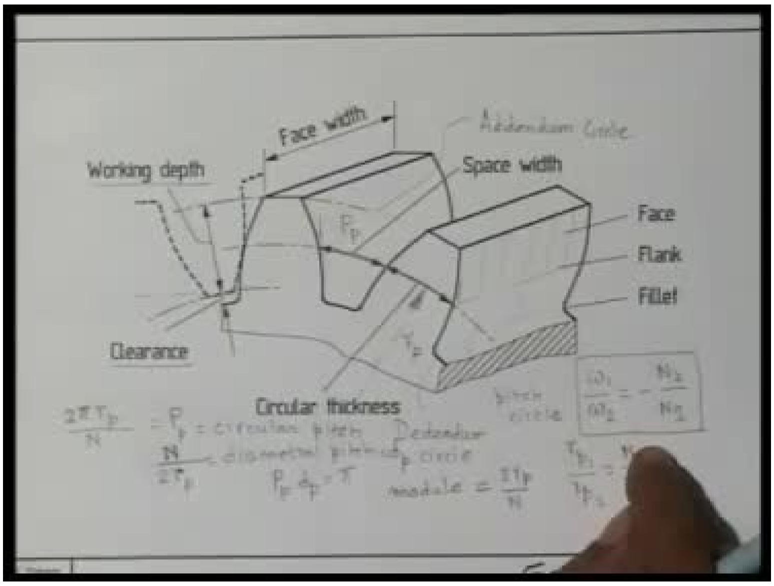
These two circles touch each other at this point, let me call this point P. As we see due to this external gearing, clockwise rotation of this gear is converted to counter clockwise rotation of this gear and the angular velocity ratio is ω_1/ω_2 is constant. Exactly the equivalent transmission is also possible by mounting two friction disks, one disk of this radius and the other disk of this radius.

Imagine two friction disks with center here and radius this much and another centered here and with radius this much. Now if you imagine these two friction disks, then one can drive the other without slip, then the same angular velocity ratio is transmitted. That means, if I call this radius say r_{p_1} and this radius r_{p_2}, then ω_1/ω_2 is nothing but $-r_{p_2}/r_{p_1}$. Because there is no slip, so at this contact points their velocities are same, so $r_{p_1}*¿$ ω_1 = $r_{p_2}*¿$ ω_2, but ω_1 and ω_2 are in opposite direction, that is taken care of by this negative sign. These two imaginary circles do not exist in gear are called pitch circles and as we see most of the gear tooth geometry will be referred to these imaginary pitch circles.

Now, let us imagine, that this r_{p_2} is made larger and larger and ultimately r_{p_2} tends to infinity, if r_{p_2} tends to infinity, then this pitch circle is converted to a straight line and such a gear whose pitch circle radius is infinite is called a rack and this is the pinion which can also engage the rack, where this r_{p_2} is infinity and we have a straight line instead of this pitch circle.

Such a rack and pinion arrangement are very frequently used to transmit, constant speed from circular speed to linear speed or linear speed to circular speed, that means a rack and pinion is very often used for conversion of uniform angular motion to uniform rectilinear motion or the other way around. If we drive the pinion then the rack will have uniform linear motion and if we drive the rack then the pinion will have constant angular motion. So, for conversion of uniform rotary to linear or linear to rotary motion can be achieved by using a rack and pinion arrangement.

(Refer Slide Time: 13:35)



Let me now, define different terms of gear tooth geometry with reference to this figure. We have shown two adjacent teeth of a gear and one tooth of the mating gear. These dotted lines show the tooth of the mating gear which is in contact with this gear, of which we have shown two consecutive teeth.

This circle is the pitch circle of this gear. This width along the axis of the shaft, that is called face width of the gear and this circumferential distance measured along the pitch circle between two adjacent teeth - the right-hand face and the left-hand face, this is called space width.

This surface where the contact takes place as it is taking place here, that is called the face of the gear. On the face of the gear, the portion about the pitch circle, that is called face and below this pitch circle, this is called flank and as you see, there is a little fillet to connect to the body of the blank of the gear. If we imagine an outer circle passing through the top outer most surface of this gear, that circle is called addendum circle.

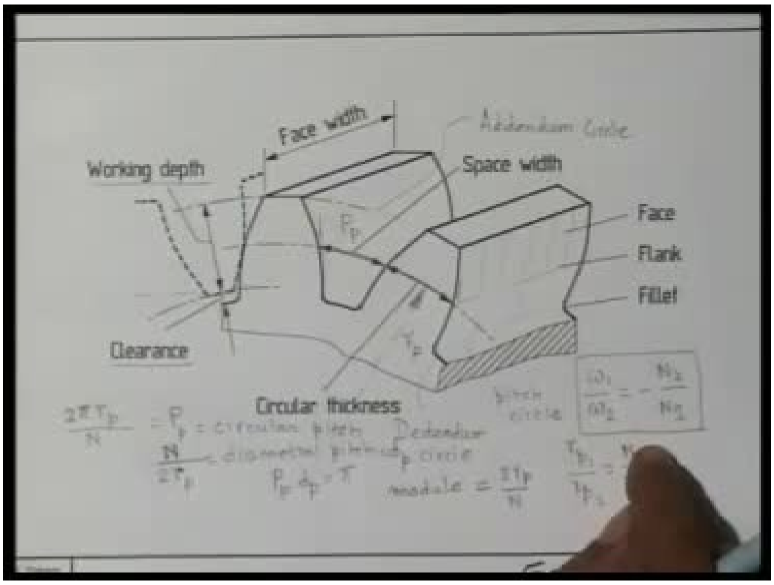
Similarly, this bottom circle which belongs to the body of the gear when the teeth ends this is called the dedendum circle. The thickness of this tooth, as you see varies from the bottom to the top, if this thickness is measured along the circumference of the pitch circle, that is called circular thickness at the pitch circle. The circular thickness keeps on varying, it is maximum here and here it is minimum. As you see we must prevent the rubbing of this top surface of the mating gear with the bottom surface of this gear. This is the addendum circle of the top gear, is the top surface from the center of that gear, if you draw a circle that would addendum circle of the mating gear and it is this distance if measured along the radius of this circle that is called clearance.

The radial clearance between the addendum of the mating circle and the dedendum of the other gear is called clearance. And if we measure the radial distance between this addendum circle of gear 1 with the addendum circle of the mating gear, that is this distance; this circle refers to the top surface of this tooth, this circle refers to the top surface of this tooth, the mating tooth and the radial distance between them is called working depth.

If we measure the distance along the identical points of two adjacent teeth of the same gear and measured along the pitch circle that is from here to there; this is one point on this tooth, I go to the identical point of the next tooth that is here, then this distance measure along the pitch circle is called circular pitch.

If I call it P_P, distance measured along the pitch circle, then P_P is called circular pitch. The radius of this pitch circle, we have already denoted by r_p.

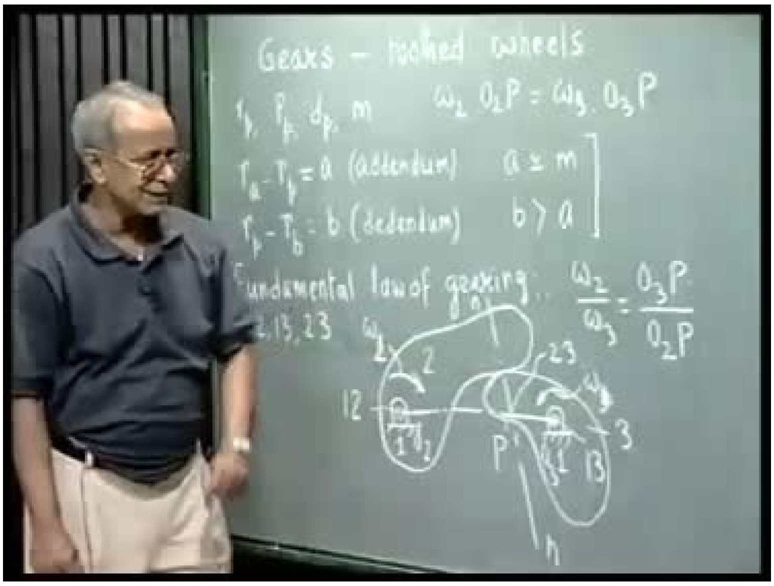
(Refer Slide Time: 19:50)



We also define something called diametral pitch; it means the number of teeth per unit length of the pitch circle diameter. If the number of teeth is N, then $N/2r_p$, that is number of teeth by the pitch circle diameter (since, $2r_p = d_p$) is called diametral pitch. And what is this circular pitch, it is easy to see that this total perimeter of this pitch circle will be $2\pi r_p$ and if there are N teeth on this gear, then $P_P = 2\pi r_p/N$, where N is the number of teeth. From these two relations, if we multiply P_p/d_p, as we see N and 2π cancels, we get $P_p \times d_p = \pi$. The inverse of the diametral pitch, which is more commonly used these days rather than the diametral pitch is called module of the gear tooth and all mating tooth have the same module; module is nothing but inverse of the diametral pitch that is $2r_p/N$, where r_p is the pitch circle radius and N is the number of teeth.

Which clearly tells that this pitch circle radius is proportional to the number of teeth for a pair of mating gears, because module is same. That means, r_p by N is constant, so $r_{p_1} / r_{p_2} = N_1/N_2$. We have already seen the angular velocity ratio $\omega_1/\omega_2 = -\ r_{p_2} / r_{p_1}$, which means, $\omega_1/\omega_2 = -\ N_2/N_1$ for a pair of external gears. So, the angular velocity ratio is inversely proportional to the number of teeth in the gear, with a negative sign for external gearing and with a positive sign for internal gearing.

(Refer Slide Time: 23:40)



We have given so far various definitions, which define the tooth geometry of circular gear like pitch circle radius, then circular pitch, then diametral pitch, then module. We have also defined addendum circle and the difference between the radius of the addendum circle and the pitch circle is called addendum, we indicate it by the symbol ‘a’; ‘a’ is called addendum.

Similarly, we defined the dedendum as $r_p - r_b$, where r_b is the dedendum circle radius, we call this as ‘b’ and ‘b’ is called dedendum. This ‘a’ and ‘b’ everything is finally standardized because gears are used universally. So, they are standardized according to various standards and ‘a’ is normally of the order of ‘m’ and ‘b’ is greater than ‘a’. Typical values of the standards will discuss much later. What we should know that to have that clearance ‘b’ should be greater than ‘a’; and ‘a’ is of the order of module.

Next, we should discuss what should be the tooth profile such that the constant angular velocity ratio is maintained. There is a line contact between a pair of teeth of the mating gears, so we have a higher pair. Now, what should be the tooth profile, such that constant angular velocity ratio at every instance is maintained. This is what we call fundamental law of gearing. To discuss the fundamental law of gearing, let say one gear as its revolute pair with fixed link here and the other gear is mounted there, that is the revolute pair of the second gear mounted to its gearing.

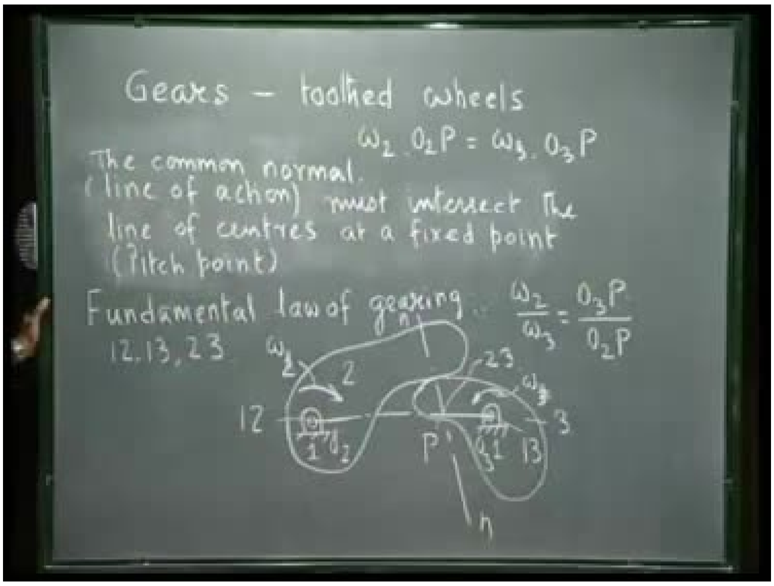
Now the question that is asked, suppose this is one gear and this is another gear, what is the condition necessary, such that these two gears maintains a constant angular velocity ratio. So, let me say this is fixed link 1, this is gear body 2, this is another gear body 3. So, this revolute pair obviously is the relative instantaneous center 12 and this is the relative instantaneous center 13. Now the question is, where is 23? We do not know where is 23 but definitely lies on the common normal. If I draw this common normal, then 23 lies on this line nn. Applying Arnhold - Kennedy theorem, we know that 12, 13 and 23 must be collinear.

So if you draw the line 12, wherever it intersects this line nn, this must be the location of 23. Now, if this gear rotates in these direction, at this instant angular velocity is ω_1 and this gear rotates with angular velocity ω_2, let me call this point of intersection 23 as P. The velocity of this point considers to be point on body 2 must have the same velocity if I consider this point P to be a point on body 3. If this point I call O_2 and this point I call O_3, O_2 is 12, O_3 is 13. So O_2P into ω_2, sorry this I call ω_1, let me call it ω_2 and this let me call ω_3, because body 1 is the fixed body, I am representing one of the gears as body 2, so I denote it by ω_2 and the other gear is body 3, so I call ω_3. So, ω_2 ¿ O_2 P in this direction must be same as ω_3 ¿ O_3P.

We consider the velocity of the point P to be a point on body 3, then its velocity due to this ω_3 is again in this direction which is perpendicular to this line O_2O_3, which is ω_3 ¿ O_3P.

Here I have measured ω_2 in clockwise and I have measured ω_3 in counter clockwise, so there is a negative sign involved here. $\omega_2/\omega_3 = O_3P/O_2P$. Now, if ω_2/ω_3 wants to maintain constant, O_2 and O_3 are fixed point, then this point P also must remain fixed, then the shape of the body should be such that the common normal always passes to the same point P on the line O_2O_3; this line O_2O_3 is call the line of centers and this line P is call the line of action which is the common normal between the mating surfaces.

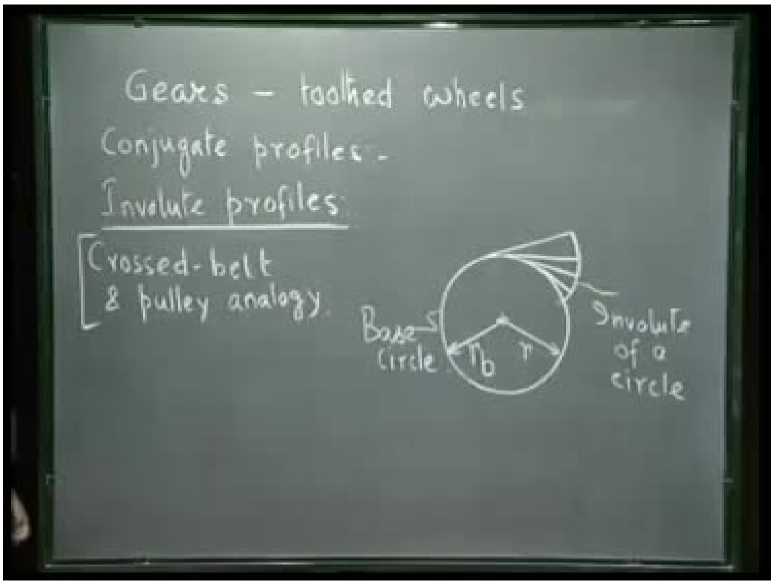
(Refer Slide Time: 30:48)



So, for ω_2/ω_3 to remain constant, this point P must not move, this point P must be fixed point on the line O_2O_3 and this relationship is called fundamental law of gearing, which in words, I can express as the common normal, which we call line of action must intersect the line of centers, that O_2 is the center, O_3 is the other center of the gear, then only the constant angular velocity ratio is maintained and this is known as fundamental law of gearing.

This fixed point called as pitch point. Next, we shall discuss there are various profiles which are possible to maintain this fundamental law of gearing and maintain this angular velocity ratio, but as we know, most commonly it is the involute profile which is used for the gear tooth.

(Refer Slide Time: 32:28)



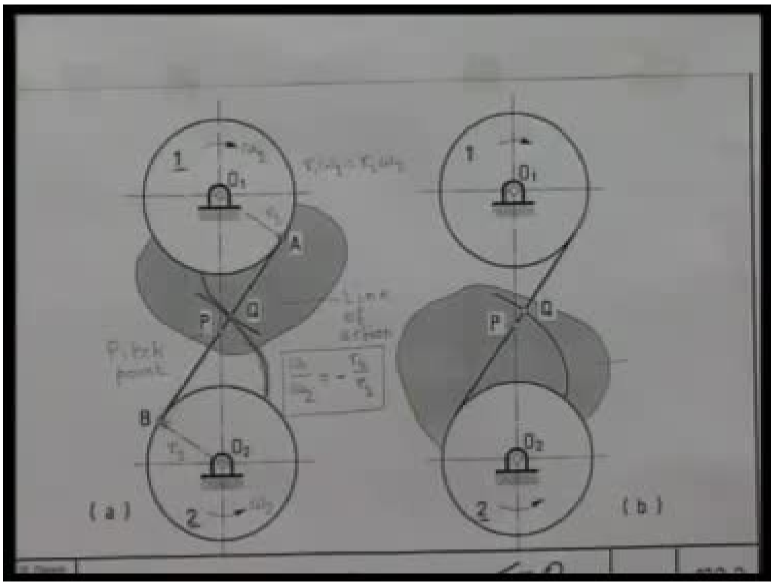
Now, let me see, what is an involute? As I said just now gear tooth profile should be such that they should satisfy this fundamental law of gearing. Actually, these profiles which satisfy the fundamental law of gearing are called conjugate profile. In fact, if one profile is given, whatever the arbitrary so long as it is continuous, we can always find another profile which is conjugate to the given profile such that the fundamental law of gearing is satisfied.

But in practice we use involute profile. Most common gear teeth are of involute profiles. As we will discuss subsequently this involute profile have lot of advantages including the manufacturing facility (fabrication advantage), convenience, so we normally use involute profiles.

Now what is an involute? Involute of a circle is defined as follows; let us take a circular cylinder say of radius r and a string or tape which is wound around this cylinder; this is the end of the tape and now if I unwound this tape from the cylinder, keeping this tape or the string always taut, that means, this portion of the tape which has been unwound, which was originally on the surface of the cylinder and now it is being unwound.

So this is the tape at one position, this is the tape at another instance, this is the tape at another instance. At every instance, the tape will be tangential to this original cylinder because it is kept taut, then the end of this tape, the curve that it generates is called involute of a circle, because initially the tape was wound on a circle. Thats why we can define involute of any curve, that means, the originally the string was wound on that particular curve, but we are not interested in anything but the involute of a circle and this circle from which the string is being unwound is called base circle of this involute and we shall denote the radius of the base circle by r_b and this is involute of this base circle of whose radius is r_b.

(Refer Slide Time: 36:14)



Now let us show what we call a crossed belt and pulley analogy, to explain why this involute profile will satisfy the fundamental law of gearing or a pair of involute will be conjugate profiles. As I said just now, let me now explain how a pair of involute tooth profiles can maintain conjugate action, that is, they can transmit constant angular velocity ratio satisfying the fundamental law of gearing. This we shall do by an analogy with a pulley and crossed belt drive. Let's say this is pulley 1 which can rotate about the point O_1; similarly, this is again another pulley, which is pulley 2, which can rotate about this axis at O_2. This common tangent between these two circles representing a pair of pulleys represents the belt, one side of the crossed belt. You can imagine another common tangent on this side, which will complete the crossed belt; so, this is a crossed belt overlapping these pair of pulleys.

Suppose, the pulley 1 rotates in the clockwise direction with angular speed ω_1, if we assume there is no slip between the belt and this pair of pulleys, then the speed of any point of the belt is given by $r_1\omega_1$ in this direction, where the r_1 is the radius of pulley 1.

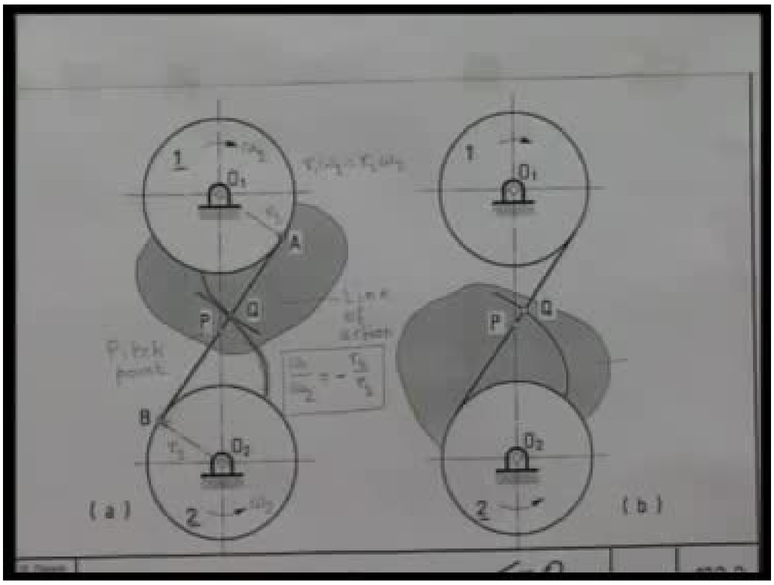
Because there is no slip between the pulley and the belt, the angular velocity of this pulley, ω_2 will be such that the speed of the belt is, also equal to $r_2\omega_2$, where r_2 is the radius of pulley 2. So this crossed belt and a pair of pulleys transmit angular velocity ratio between these two bodies, namely 1 and 2, where ω_1/ω_2, if we take care of the sign, this is clockwise, this is counter clockwise. So, from this relation we get $\omega_1/\omega_2 = -r_2/r_1$. At this stage, let me imagine a pencil connected to a particular point on this belt, say at this point Q; if I attach a pencil here, as this belt moves, this particular pencil will draw this straight-line AB, it will start from this point; this point Q sometime before at A and as the belt moves the pencil moves along with the belt and draws this line AB, in this piece of paper or in the fixed space.

Let us now try to imagine, if I attach a body which is integral to pulley 1, this is an extension of pulley 1 and this is same rigid body. What is the curve drawn by the same pencil, which was attached to this belt, on this body, that is here. To determine that particular curve which is drawn by the pencil on body 1, we will hold the pulley 1 fixed; if we don't allow the pulley 1 to rotate, then this string which is unwinding from this pulley unwinds in the counter clockwise direction because the pulley was rotating in clockwise direction and the belt was straight. Now, if we hold the pulley fixed, then the belt as it unwinds rotate in the counter clockwise direction. So as a result, this point Q on this belt or string generates an involute of this pulley, that is this circle represents the base circle of this involute.

This particular point when Q was here, this particular point of the pulley was there. When the Q has move down by the distance AQ, this point on the pulley, because pulley rotating in the clockwise direction has move there. So, the same pencil which is drawing this line AB in the space or in the piece of paper is drawing this particular involute on this body 1. This circle represents the base circle of that pulley, base circle of this involute and this is the string which is generating the involute.

Exactly using similar logic, we can say the same pencil attached to the Q will generate this involute on this body. This is an integral part of pulley 2, this is rotating in the counter clockwise direction; so, if we hold body 2 fixed, then I see the string which is winding on this particular cylinder should wind up in the clockwise direction.

(Refer Slide Time: 41:44)



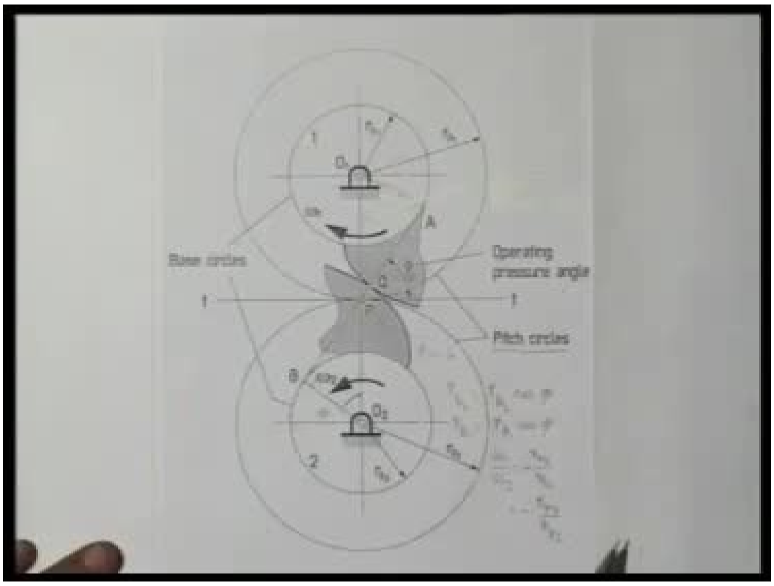
Because this was not moving, this was not having any rotation; this was rotating in the counter clockwise direction. So if I hold this fixed, this winds up and also rotates in the clockwise direction and generates this is involute. Now, if we cut this body along this profile and this body along this profile, then this involute we can have here. Now the point of contact between these two involutes is always on this line and if we remove this belt now and rotate this body 1 in the clockwise direction; as we see this involute will push this involute in this direction and this will rotate in the counter clockwise direction; maintaining exactly the same relationship as it was doing when they are connected by this belt.

So, exactly identical motion between this pair of pulleys with the crossed belt can be achieved by cutting an involute profile on body 1 and an involute profile on body 2 and allowing one involute to drive the other involute; again, $\omega_1/\omega_2 = -r_2/r_1$.

Thing to note is that, as this body rotates the point of contact changes on both these profiles, but it is always lie on this line AB. A further clockwise rotation will may counter clockwise rotation, this point may come in contact with this, but both of this point will lie on this line somewhere. So, this line AB which is normal to these involutes because this is the string, this is an involute, so at every instant the string is perpendicular to the involute. So, this line AB which is common normal between these two involute profiles defines what we call the line of action and this line of action AB intersects the line of centers at this point, fixed point P.

So, let me repeat, these are the two involute profiles which maintains the same motion as was been done by this crossed belt; if you remove the crossed belt and allow this involute to drive this involute, then this constant angular velocity ratio is maintained. The point of contact moves along the line AB which we call the line of action and this line AB is the common normal at the point of contact to the pair of involute. This is the line of action and this point P we call, pitch point. Finally, let me complete the sentence, using this analogy between pair of pulleys given by a cross belt, we have shown that pair of involute profile can maintain the same conjugate action, that is constraint angular velocity ratio even if we remove the belt.

(Refer Slide Time: 45:33)



Because a point on the belt which is generating this line AB in space, is generating a pair of involutes on body 1 and body 2. Let me continue discussion with belt-pulley analogy, as we have seen that this is pulley 1 of base circle radius r_{b_1} ; this is pulley 2 which is the base circle radius r_{b_2} ; this AB is the common tangent which represented the belt and these are the two involute profiles, which as I said can be machined on body 1 and body 2 and remove the belt and rotation ω_1 through this involute profile will cause rotation angular velocity ω_2 of body 2, through this contact between these two involute profiles. By the very construction, if this is the string which is always taut and this point is generating this profile, it is obvious that this line AB is normal to both of this involute profiles. That is, this common tangent AB between these two base circles is the common normal to these involute profiles, which means, this is what we call line of action which is the common normal between the profiles.

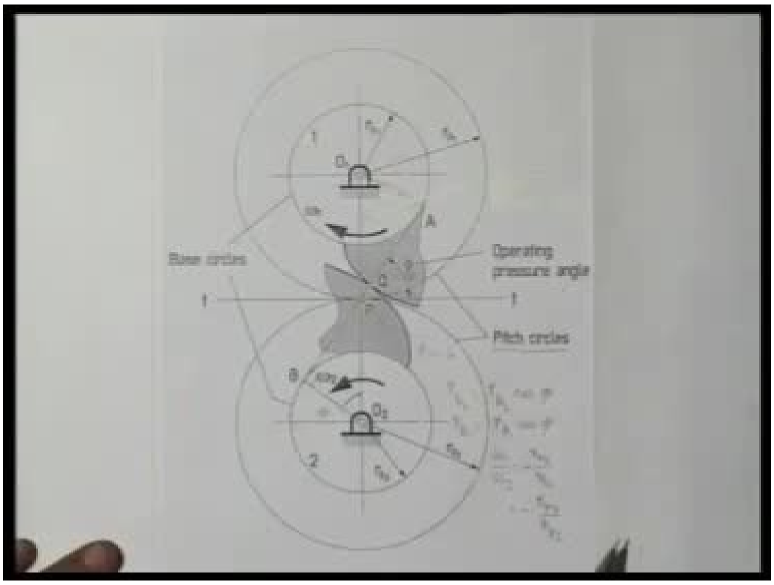
This common normal is intersecting the line of centers O_1O_2 at this point P and as we see because of these base circles, this point P never changes as these two belt rotates because this always remains the line of action, the common normal when the contact is here and here, anywhere at this point. The contact is shown as the gear rotates the contact changes from here to there, but this common normal always remains the same.

It is this common normal which intersects the line of centers O_1O_2 at this point P which is a fixed point, which is not changing. So, this point P, we call pitch point. So, this clearly establishes that these two involute profiles maintain the conjugate action because the pitch point is fixed, it does not change on this line O_1O_2, common normal AB intersects O_1O_2 at this fixed point P. If I draw these two circles with O_2P and O_1P as radius; these and these of radius r_{p_1} and radius r_{p_2} are pitch circles.

The angle that this common normal AB makes with the common tangent to these two pitch circles, tt is the common tangent to these two pitch circles of radius r_{p_1} and r_{p_2}. The angle that this common normal AB to the base circles makes with this common tangent to the pitch circles is called operating pressure angle (φ).

It is obvious if this angle is φ and then this angle is also φ; similarly, this angle is also φ. O_1A is the radius and this is the point of common tangency A and B. So, angle between O_1A and the vertical line is same as the angle between AB and this horizontal line. So, this is φ, pressure angle; this is also φ, the pressure angle and again this angle is also φ, the pressure angle. So, what we see that r_{b_2} is nothing but r_{p_2} cos φ, because this angle is 90^0, angle between the tangent and the radius is 90^0. Similarly, this angle is 90^0, angle between the radius and the tangent and this is r_{p_1} and this is r_{b_1} and the angle between them is φ; so, what we get $r_{b_2}=¿$ $r_{p_2}\cos\varphi$, where φ is called the operating pressure angle and $r_{b_1}=¿$ $r_{p_1}\cos\varphi$. So the angular velocity ratio which we earlier have seen as $\frac{\omega_1}{\omega_2}=\frac{-r_{b_2}}{r_{b_1}}=\frac{-r_{p_2}}{r_{p_1}}$, because if you divide this by this, cos φ cancels and we get this.

(Refer Slide Time: 50:01)



We have defined operating pressure angle, the pitch point, the line of action for these two involute profiles maintaining conjugate action. In our next lecture, we shall discuss what are the advantages of such involute profile.