Discrete Mathematics
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Lecture-39
Solving Recurrence Relations (Part 2)

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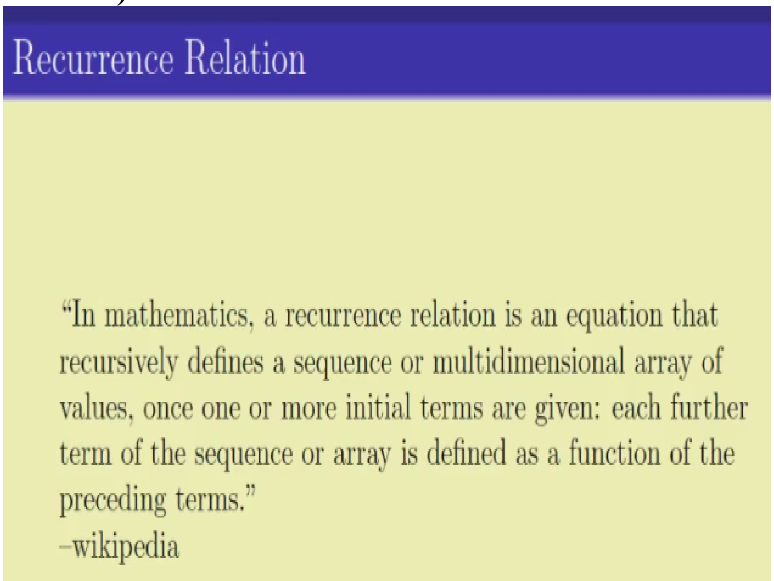

Welcome back, in the last few videos, we have been studying recurrence relations. So recurrence relations is basically an equation that recursively defines a sequence of values. There are some initial terms and the n^{th} term is defined as a function of the preceding terms. Recurrence relations have been used extensively for combinatorics, analysis of algorithms, in computational biology, in theoretical economics and in various other subjects.

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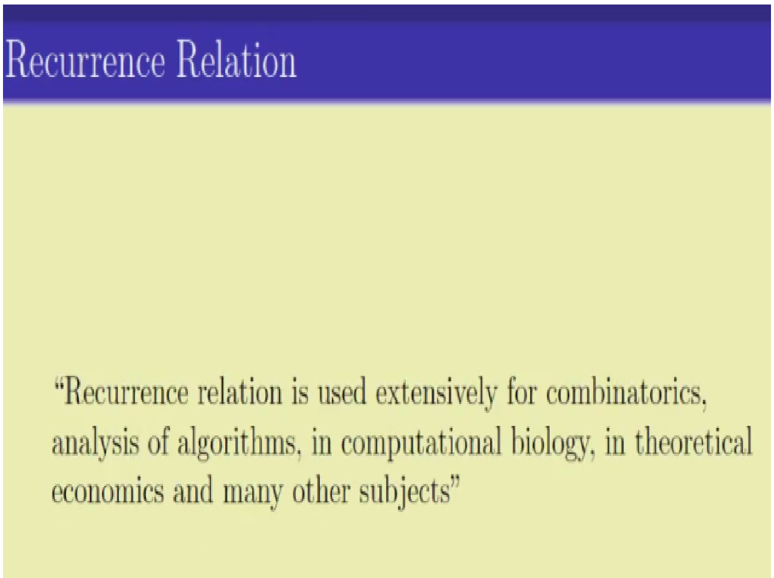

In the last couple of videos, earlier we saw how to use the recurrence relations for modelling some of the counting problems.

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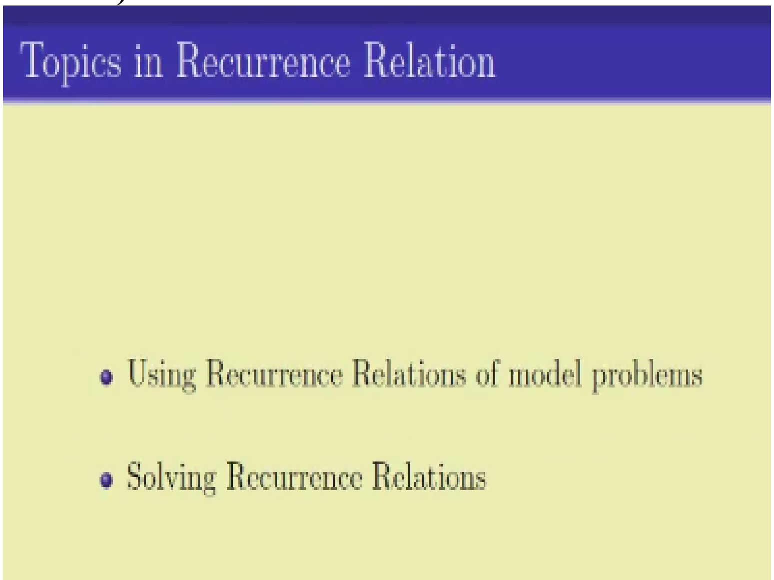

Now once you model some of the counting problems, you have to solve the recurrence relations in some way. So here some of the examples that appears in real life, say for example, T(1) = 1 and T(n) = 2 + T(n-1) or T(1) = 2 and T(2) = 3 and T(n) = T(n-1) + T(n-2) or this is the one that came from the Tower of Hanoi problem.

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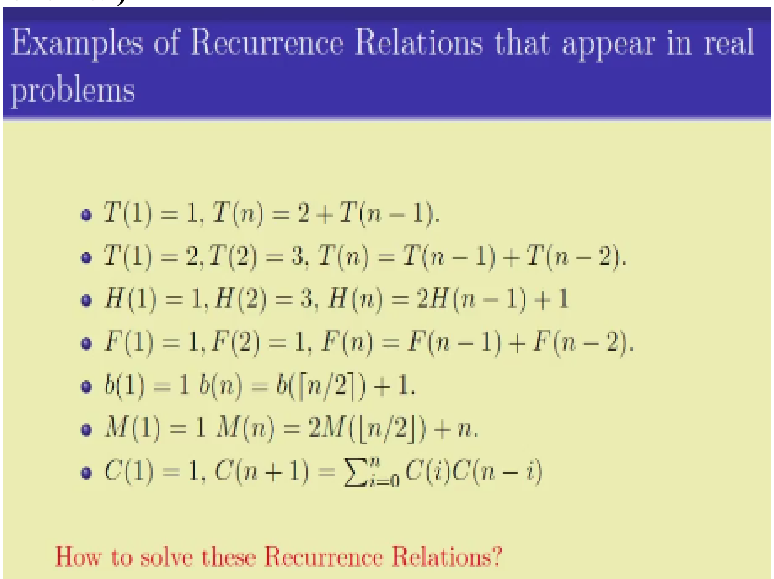

H(1) = 1, H(2) = 3 and H(n) = 2H(n-1) + 1, or this is the one that come from the Fibonacci sequence, F(1) = 1, F(2) = 1 and F(n) = F(n-1) + F(n-2), or this one that comes from the binary search algorithm, b(1) = 1 and $b(n) = b(\lceil n/2 \rceil) + 1$, or this one that comes from the merge sort algorithm M(1) = 1 and $M(n) = 2M\left(\lfloor \frac{n}{2} \rfloor\right) + n$, or this one which comes from, what is known as Catalan number C(1) = 1 and $C(n+1) = C(i)C(n-i)$ C

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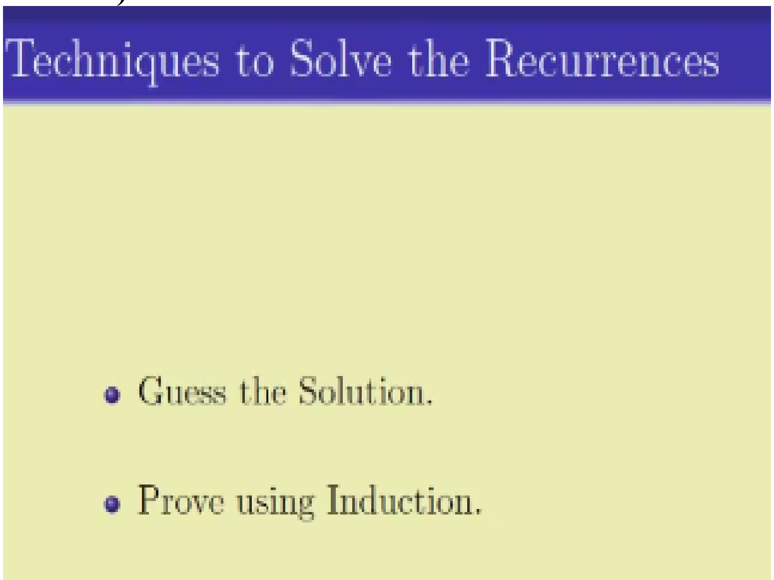

Now these are some of the recurrence relations that appear in real life, these are the very small sample of them. Now the main question is how do you solve these recurrence relations? So these recurrence relations are ofcourse used to module various problems, but once you model them into a recurrence relations, the next step is to solve them. In the last video, we saw a technique of solving them.

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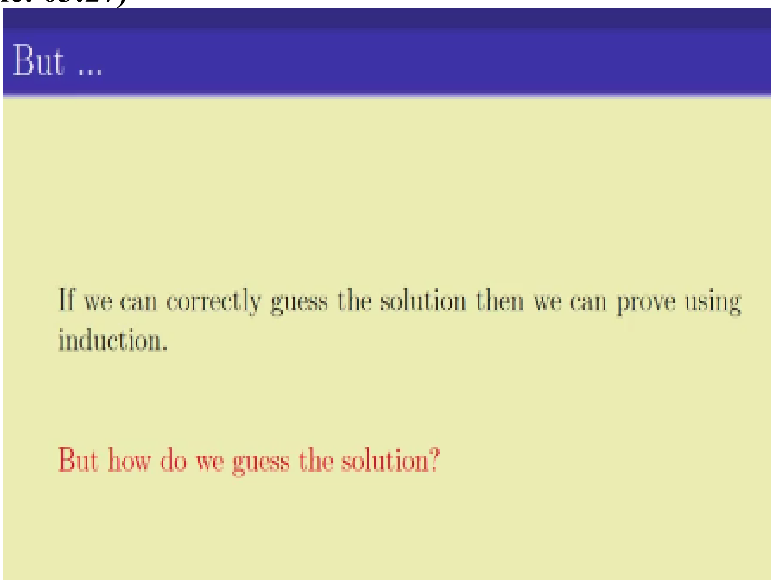

And we told these are the techniques that first of all guess the solution and then proves using induction. We saw that if we can guess the solution correct, then proving it by induction possibly not to hard a problem, it is like the typical induction problem. The main issue is how do you guess the solution? Now guessing the solution can really be a challenging problem. So we will be dedicating quiet a number of lectures on guessing the solution.

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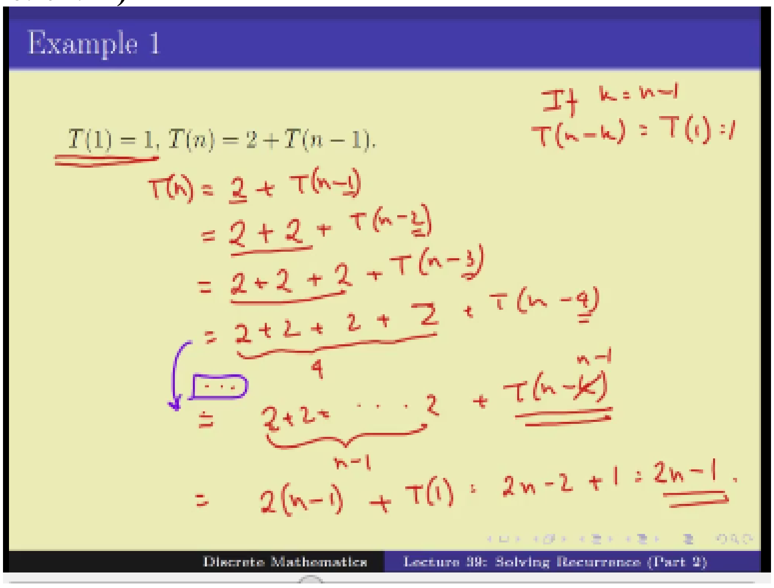

Today we will be looking at the first and the simplest technique of guessing the solution. So here it is, so technique one, the idea is just unfolding the definitions. What do I mean by unfolding the definitions? So let us look at some of the examples and you will understand what do I mean by that? Note that these are not formal proofs, these are mere guessing which might work or might not work and whether it works or not, of course you have to go back to the induction and prove it and only then we get a formal proof.

So this is whatever I am going to say it now is how to guess this step? Say if T(1)=1 and T(n)=2+ T(n-1). Now we can recursively open up terms as follows:

T(n)=2 + 2 + T(n-2) = 2 + 2 + 2 + T(n-3) = 2 + 2 + 2 + 2 + T(n-4)

So if I have keep on doing this way, I will get

T(n) = 2 + 2 + ... 2 (*k terms*) + T(n-k)

Now this is a leap of faith. Okay, again as I told it is a guessing work. Now this number (T(n-k) is somehow we have to vanish. The idea is that, the initial things, here T(1)=1, gives us the hint, so we have to set, so here if k = n-1, then T(n-k) = T(1) = 1. So I get

T(n) = 2 + 2 + ... 2 (*n-1 terms*) + T(n-(n-1)) = 2(n-1) + T(1) = 2n - 1

So this by doing so we have guess that T(n) =2n-1, again also it might seem very formal way of proving that T(n) =2n-1, the fact is that it still not a correct proof, a complete proof, because we have from initial terms to k terms, there was a big leap of faith.

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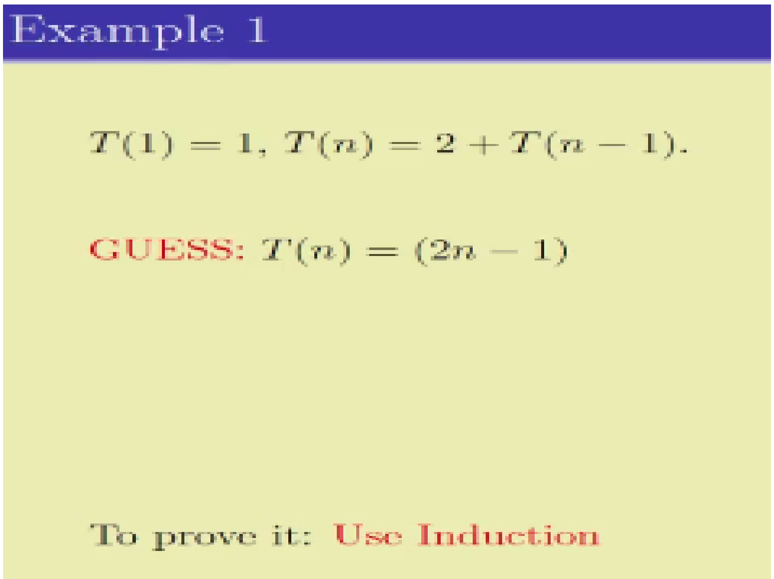

May be our intuitions are correct and we get the right answer and in which case we go ahead and prove it using induction. And there are examples where this leap of faith may not be exactly correct. But this is one way of kind of guessing what the number is. So in this case, we have T(n) = 2n-1, is the guess and you prove it by induction. We had seen in the last video that this is indeed the right guess by proving it by induction.

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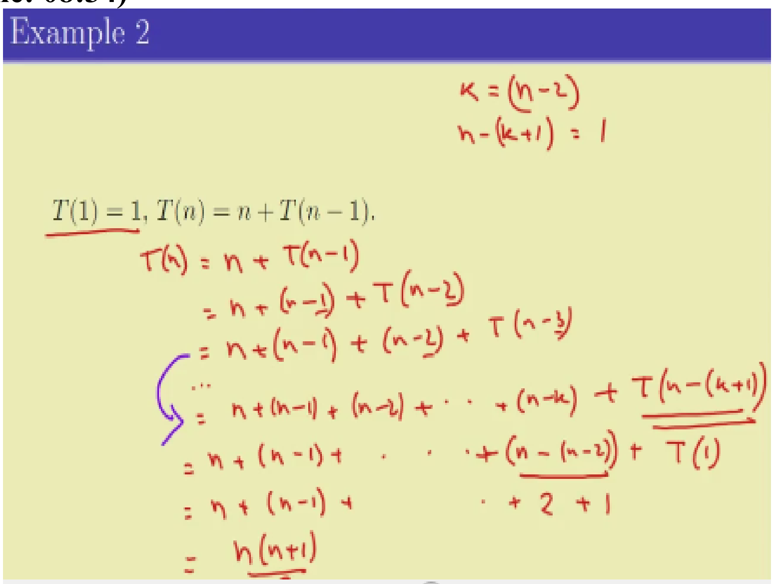

Now let us move on to the next example, so here T(1) = 1, T(n)= n+ T (n-1), again we have to keep on unfolding a definition, as follows:

T(n) = n + T(n-1)

= n + (n-1) + T(n-2)

= n + (n-1) + (n-2) + T(n-3)

.....

= n + (n-1) + (n-2) + ... + (n-k) + T(n – (k+1)) (*This is again by leap of faith*)

Now again we have to somehow disappear this term T(n – (k+1)), so the idea is again that we have to get this n-(k+1) =1, or in other words, I take k = n-2.

T(n) = n + (n-1) + (n-2) + ... + (n-(n-2)) + T(1)

= n + (n-1) + (n-2) + ... + 2 + 1 = n(n+1)/2 (*sum of first n integers*)

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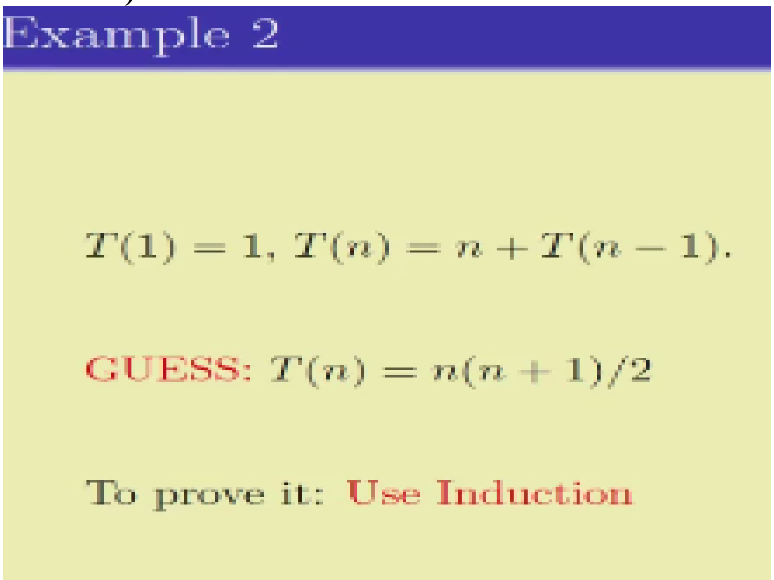

So by doing so, we have guess that T(n)=n(n+1)/2, now again as I told you this is a leap of faith, because there was a leap of faith involved and hence this is just a case, so formally proving it we have to solve it by induction and verify that our guess is indeed right. So again the simple idea is, keep on unfolding the definition and it will be possibly we are able to guess the value and in this case we did guess T(n) = n(n+1)/2 and we then prove it by induction.

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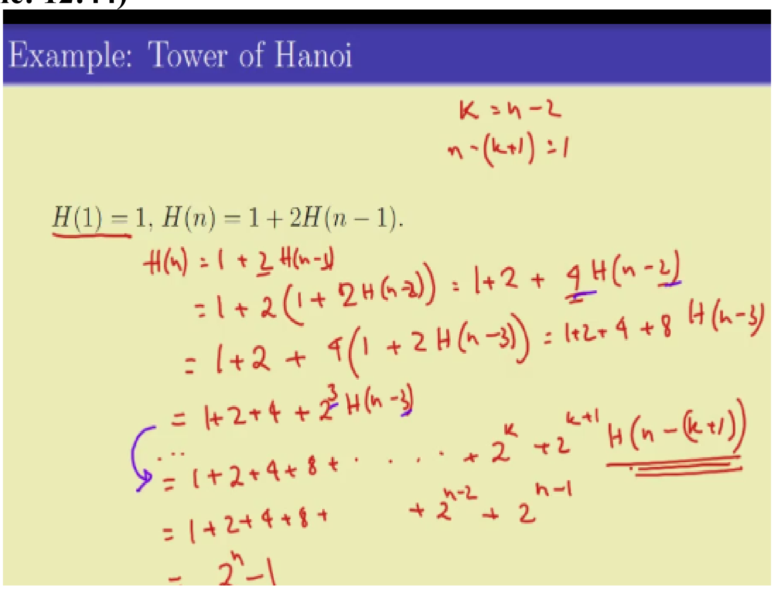

As I told you, most of the time the guess does work correctly if we can unfold it in a right way. Let us look at one more example, is the tower of Hanoi problem where $H(1) = 1$ and $H(n) = 1 + 2H(n-1)$. So by unfolding the terms,

$H(n) = 1 + 2H(n-1)$

$= 1 + 2(1 + 2H(n-2)) = 1 + 2 + 4H(n-2)$

$= 1 + 2 + 4(1 + 2H(n-3)) = 1 + 2 + 4 + 8H(n-3)$

$= 1 + 2 + 4 + 2^3H(n-3)$

....

$= 1 + 2 + 4 + 8 + ... + 2^k + 2^{k+1}H(n-(k+1))$ (*By leap of faith*)

Now here again we need to make this term, $H(n-(k+1))$ disappear, so again to make have $H(n-(k+1)) = H(1)$, take $k = n-2$., so we get

$= 1 + 2 + 4 + 8 + ... + 2^{n-2} + 2^{n-1}H(1) = 1 + 2 + 4 + 8 + ... + 2^{n-2} + 2^{n-1}$ (since $H(1) = 1$)

$= 2^n - 1$ (summation of GP)

So I guess that $H(n) = 2^n - 1$, so this one clearly was slightly more complicated than the earlier ones, but again here there was a massive leap of faith here from initial terms to k^{th} term.

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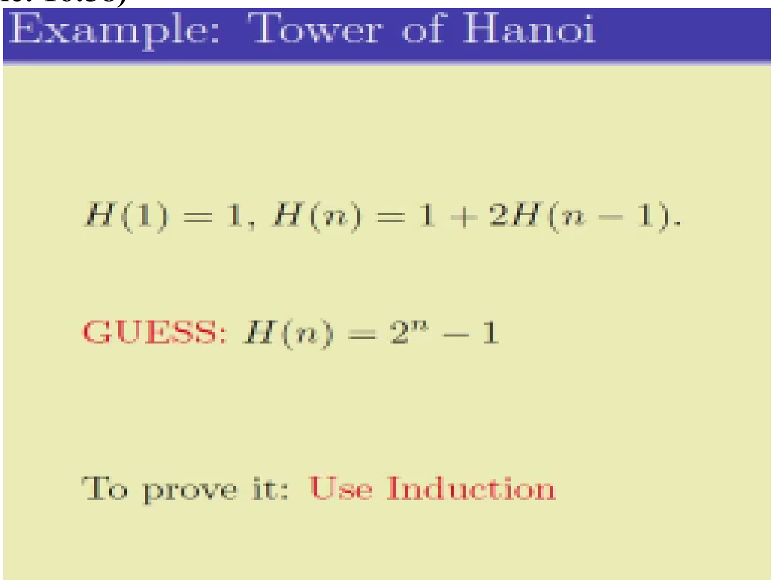

And so by doing so we have managed to guess it, but we need to prove that the guess is right again by induction. It so happens that in this case, we guess it indeed right and we saw it last time that we can guess this one and by induction we can prove the statement. So the basic idea that we learned from this video is that if have given a particular recurrence relation like this and if you can unfold it, maybe you can try to guess the number.

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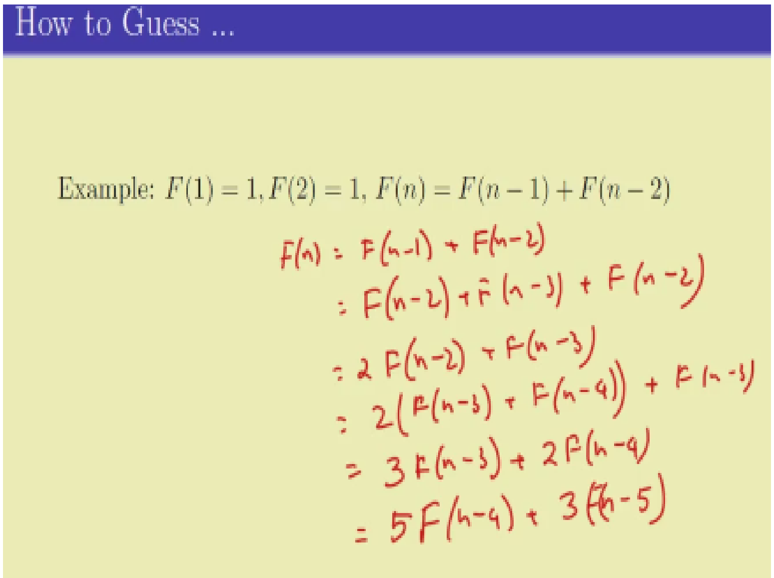

The problem is that they are complicated ones like this, F(n)=F(n-1) + F(n-2), now we can try to unfold it by saying okay,

F(n)=F(n-1) + F(n-2) = F(n-2)+ F(n-3) + F(n-2)

= 2F(n-2) + F(n-3)

= 2(F(n-3) + F(n-4) + F(n-3)

= 3F(n-3) + 2F(n-4)

= 5F(n-4) + 3F(n-5)

Of course it is clear if I keep on doing it.

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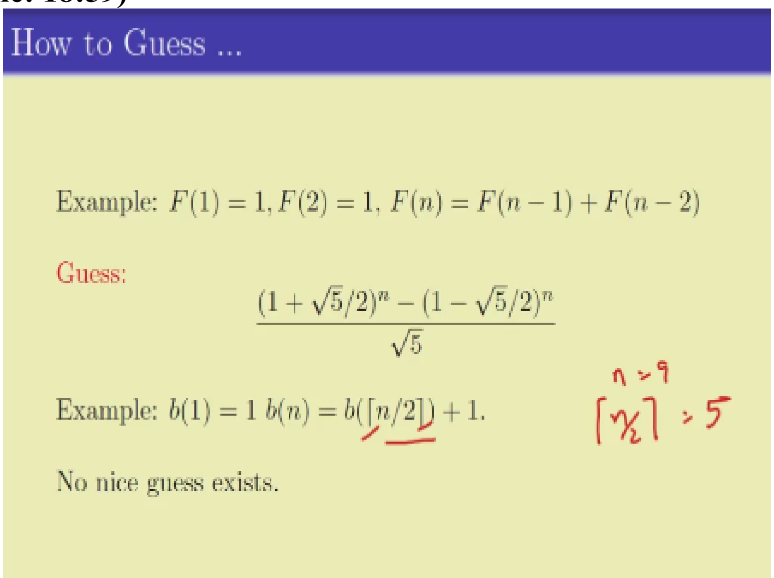

Since, particularly no pattern coming out in this recurrence relation. So in fact for this kind of recurrence relations, unfolding will not help. I leave you guys to check and verify and convince yourselves that here by unfolding you will not be able to guess the actual value, in fact guessing the actual value is quiet complicated here. Here is the actual guess

$((1+\sqrt{5}/2)^n - (1-\sqrt{5}/2)^n)/\sqrt{5}$

and you can see by looking at this expression, it is not something that is easy to guess, right. Also we have other kind of formulas like this one, where b(1)=1 and $b(n)=b(\lceil n/2 \rceil)+1$, where $\lceil n/2 \rceil$ denotes the integer that is bigger than or equal to n/2, so if n=9, then [n/2] = 5. So with this kind of an expression, unfortunately there is no clean guess can be made because of this extra bit of weird thing that are there, this what we call as ceilings right.

So there are expressions of this form where either the guessing is too hard or we do not have a very clean guessing for them. How to attack this particular kind of recurrence relation we will be doing in the next couple of weeks. Thank you.