Numerical Solutions of Ordinary and Partial Differential Equations

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Lecture - 20
Shooting Method- BVPs

Hello, good morning I hope you are comfortable by now with two point boundary value problems. And we have discussed a finite difference methods and how to solve two point boundary value problems using finite difference methods. So, but there are other certain methods for two point boundary value problems, you see we started with initial value problems, and then we have come to boundary value problems. Now, having the knowledge of initial value of problems can we use it to solve boundary value problems as well.

So, this is slightly a different technique compare to finite difference methods called shooting method. So, as the word suggests a sometimes to reach the target, sometimes we may under shoot, and sometimes we may over shoot, in the sense we may reach here above the target, sometimes below the target. So, the main idea here is with what angle we should fire, so that we reach the target, this is a kind of a intuition. So, let us a start discussing shooting method.

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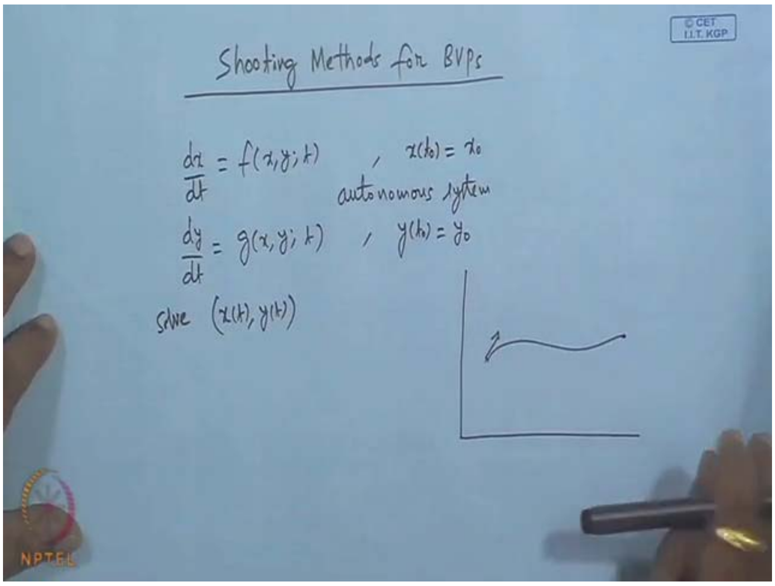

Shooting methods for boundary value problems, so let us start with some physics. For example, when we have autonomous systems say d x by d t equals to some f of x y t and d y by d t is g of x y t. So, this is called autonomous system. For example, the book by G F Siemens and ODE discusses about this. So, accordingly we must have x of some initial condition t naught is x naught y of t naught is y naught. So, that means we are looking for x of t y of t at any given time, so this is our solution right now what a shooting method does. So, we are trying to solve for so this is our let us say target, so then what is the corresponding initial angle. So, that we hit the target so let us switch over to boundary value problems.

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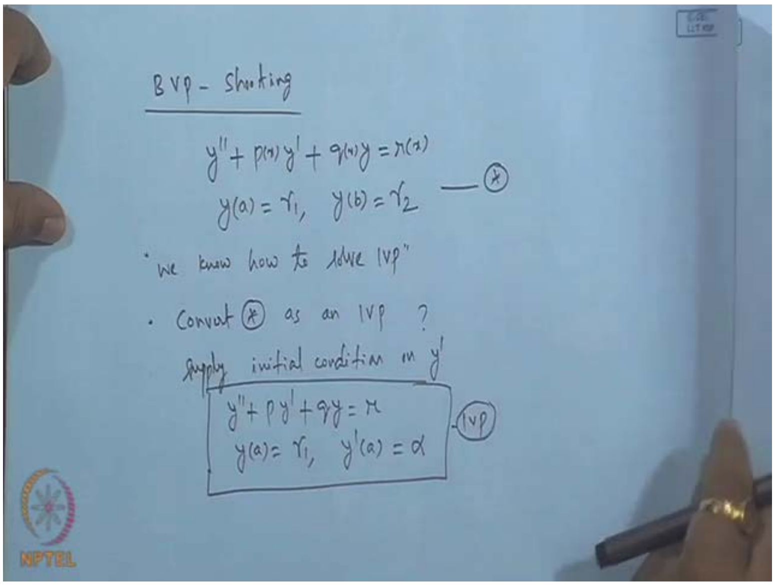

So, in the context of boundary value problem shooting, so we have say y double dash plus y dash y is r of x then we have the boundary conditions. Now, what we are trying to do is convert, so this is b v p now we know how to solve IVP. So, can we convert star convert star as a initial value problem, yes it is possible how do we supply initial condition, why we have to supply initial condition on y dash because already we have y of a, right?

So, if you supply a initial condition y dash then this boundary value problem can be converted to initial value problem. So, how do so y double plus p y prime q y is r and y of a is gamma 1 and y of y dashed of a is some alpha. Now, this is our IVP now what is our idea suppose if you solve we can get the solution at any grid point. So, what is the idea?

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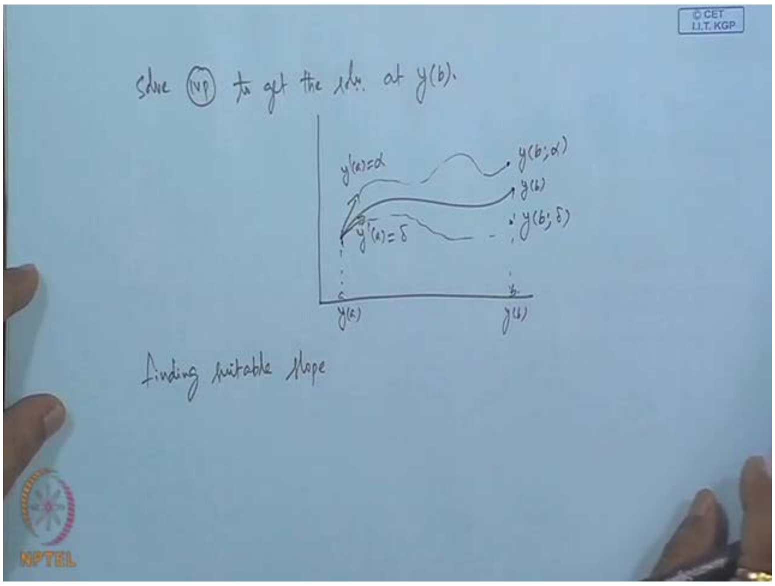

The idea is solve IVP to get the solution at the boundary point y of b, now what may happen so we know the curve let us assume the end points. So, this is y of a this is y of b, now we have started with some y dashed of a equals to alpha let us say with this we hit here. That means, something it raised and then we reach here so that means this is y at b with alpha. Suppose we slightly at just the slope, suppose next time we reach here so this is y of at b and let us say here it is y dashed of a equals to some delta. So, this is at delta, so what is happening?

If you adjust your slope you are tracing a different path, but what is our target so this is y of b, so this is y of b. So, this a and this is b so this is y of b, so we need to guess or rather we need to find out what could be the slope with which if you fire, we can reach the target. And how do we reach the target? As close as possible, so that means the problem reduces to finding suitable slope, so this is the target. So, is that clear with a shooting method now let us formulize.

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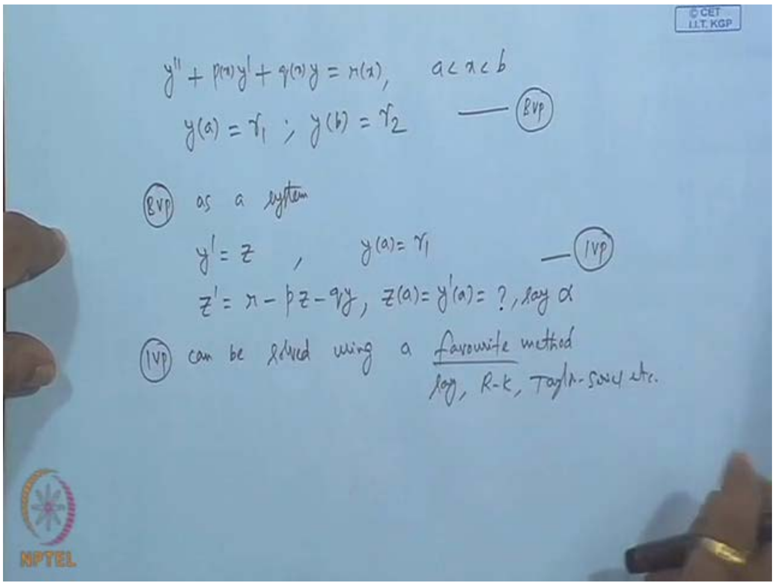

So, we have so these are boundary value problem now B v p. As a system y dash is z z dashed is r minus p z minus q y and y of a is gamma one and z of we need z of a is y dashed of a, so we would like to convert b v p as a system and if you convert we need this missing z of a. So, this is the slope, right? Now let us say alpha so then we have equivalent IVP, so we have given b v p we have converted into equivalent IVP.

Then IVP can be solved using a favorite method, so your favorite method say R-k method or Taylor series extra. So, when we solve we have a b v p we converted into equivalent IVP. However we have something to do with alpha so what we say we are saying some alpha, so we do not know what is this alpha? So, then once we convert to equivalent IVP.

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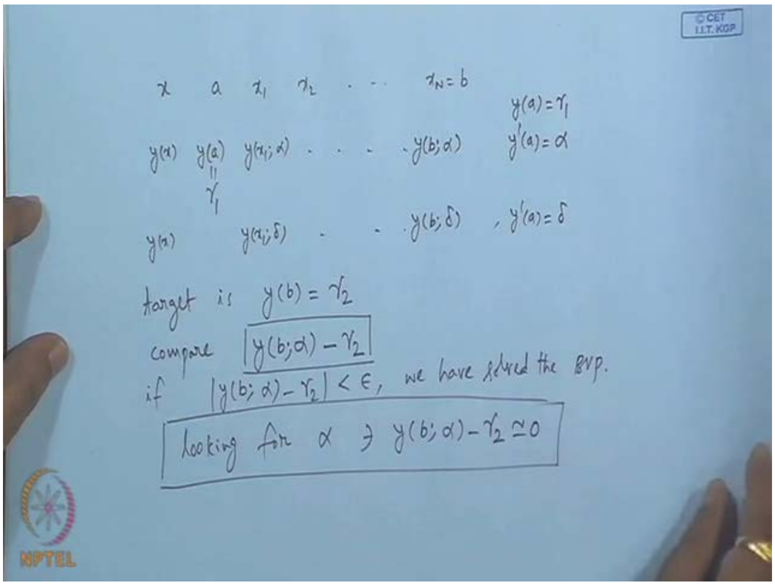

We can x say a then x 1 x 2 then x n equals to b and here we should mention our y of a is gamma one and y dashed of a is alpha. Now, if you compute so we can get the value y, y of a is given here y of x 1. Of course, with this alpha, so one can compute y of b with this alpha. Suppose, we change delta so that means this corresponds to now whatever may be the slope one chooses, what is our target? Target is y of b equals gamma 2, hence compare y of b obtained with some alpha minus gamma 2 we have to compare, if is less than epsilon then we have solved the b v p, do you get it? See we have some initial guesses alpha that is our slope so then you solve the IVP using some favorite method, we get the values at every grid point.

So, we get solution at the boundary point as well with the slope chosen, now suppose somebody has solved with alpha as a slope one of your friend's have solved with delta as a slope. Then we check the difference because our target is gamma two so the difference must be less than epsilon where epsilon is some prior signed. So, this is a desired accuracy, now with some alpha you get a let us say up to 10 power minus 2 that means up to two decimals. Suppose your friend with some other that is let us say delta gets a more than definitely your friend solution is more closer to the target. So, what is our aim? See this, what we are comparing and the condition we are looking for looking for alpha, such that so this is what we are looking for. So, that means you need to find.

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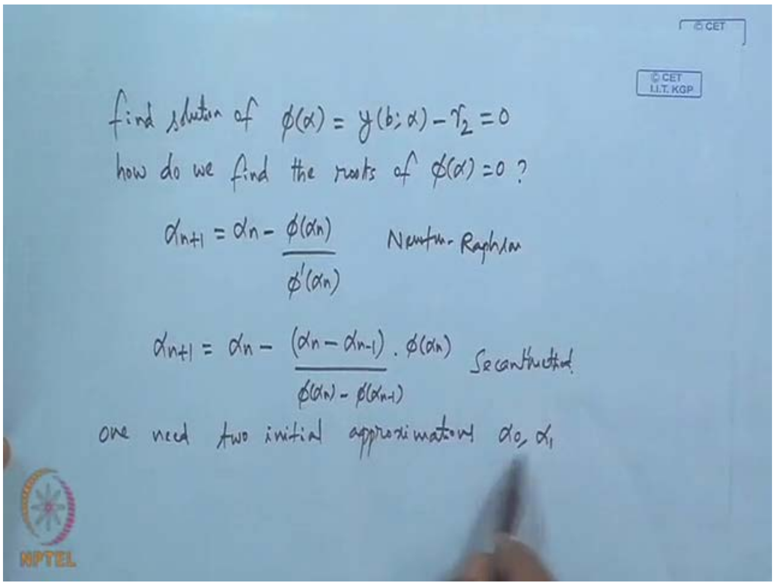

So, we are looking for this that means find the roots of this equation phi of alpha, so this we treat it as some non-linear equation, we know several methods to find the roots of this equation. So, how do we find the roots of phi alpha equals to 0, we can try we have learnt can you name some of the methods. So for example, you have Newton Raphson method any other, yes Secant method so let us define very popular Newton Raphson method. Of course, here the prime denotes derivative with respect to alpha, but the question is how do we find the derivative.

So, then let us say we switch over to Secant method, so this is secant method and this is Newton Raphson method, so to solve phi of alpha equals to 0 using secant method one need two initial approximations say alpha 0, alpha one so we need two initial approximations alpha 0, alpha 1. So, then what we do with this, so what we do we compute phi of alpha and phi of alpha and minus 1. So, when we have these initial approximations, that means these two are two different slopes.

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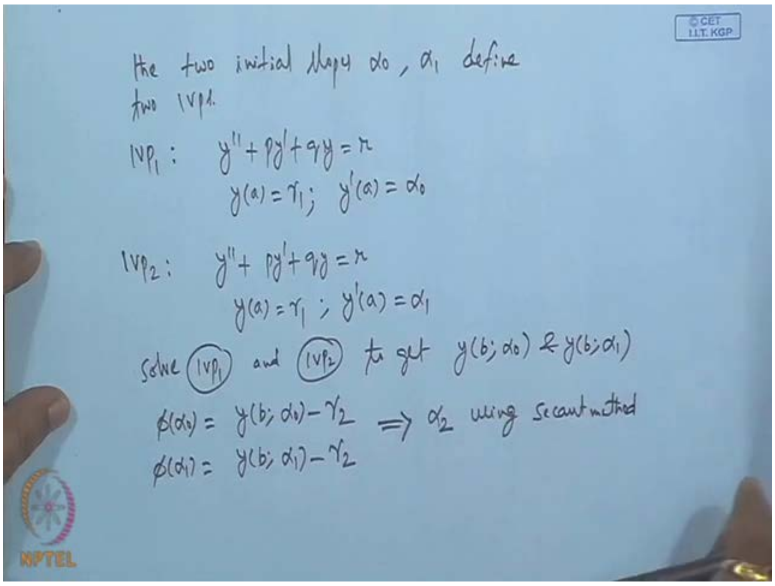

So, the two initial slopes alpha 0 alpha 1 define 2 IVP s, so IVP 1 y double plus p y prime q y is r y of a is gamma 1 and y dashed of a is alpha 0 IVP 2 IVP 2. So, we have two different IVP's then what we do solve IVP 1 and IVP 2. Solve IVP 1 and IVP 2 to get y at b using alpha 0 and y at b using alpha 1. So, then when we know this what will be phi of alpha 0 and phi of alpha 1.

So, that means with this initial slope if you start what, what would be the value at the boundary point. So, this is what we get then how far we are from the actual target because gamma 2 is our actual target. Then similarly, with this slope how far we are so once we know this we can compute alpha 2 using Secant method using Secant method. So, then this will be our new slope that could be a refinement so then we obtain alpha two what do we do?

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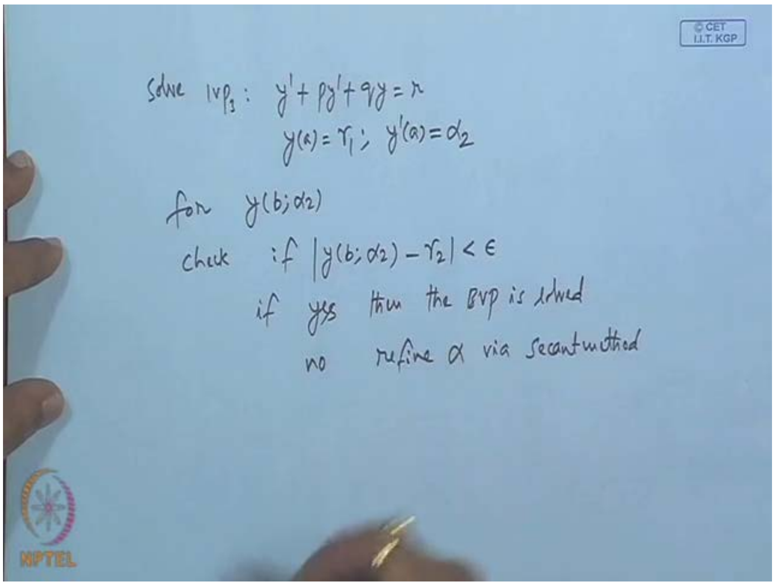

Solve IVP 3 alpha 2 then solve for what y of b alpha 2 then check if this is less than if yes then the b v p is solved if no refine alpha via Secant method. So, this is we do it so once we get some initial slopes, we start solving the IVP's then we get the value at the boundary point then compare with the target. So, let us they are far away then we refine to refine, we use secant method or we can use Newton Raphson method, but right now we are discussing secant method. So, once you refine then we get a new slope again we have to solve the IVP with the new slope to get the value at the boundary point.

So, once you get the value at the boundary point again you compare with the target that is the actual value which we are looking for and check how far we are. Suppose, we are slightly close, but we are not so happy then again we take that value and then refine using Secant method and we do this process until we are happy. That means, until the difference between the actual gamma 2, which we are looking for and the value at the end point which we have obtained via the initial value problem they differ up to decide accuracy. So, let us start with a problem shooting method via secant method.

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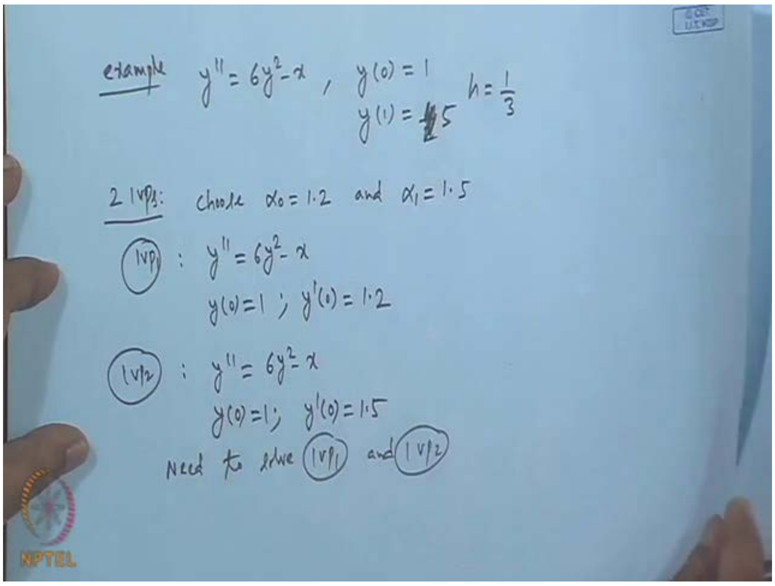

Example, suppose this is our example y of 0 is 1 and y of 1 is y of 1 is say half then h is one-third. Now, the 2 IVP is so choose, choose alpha 0 and alpha 1, alpha 0 is 1.2 and alpha one is 1.5. So, these are definitely so these initial choices etc may be some from the physical data or some experimental data. So, this is a kind of a guess, so that means we define now IVP's, IVP 1 then IVP 2, so this is IVP 2 so then y of 1, let us take 5, now we have to do? We have to solve, since for the sake of simplicity I would like to try just a Euler method please excuse me because Taylor series R-k method there little tedious, but you can try with the Taylor series R-k method.

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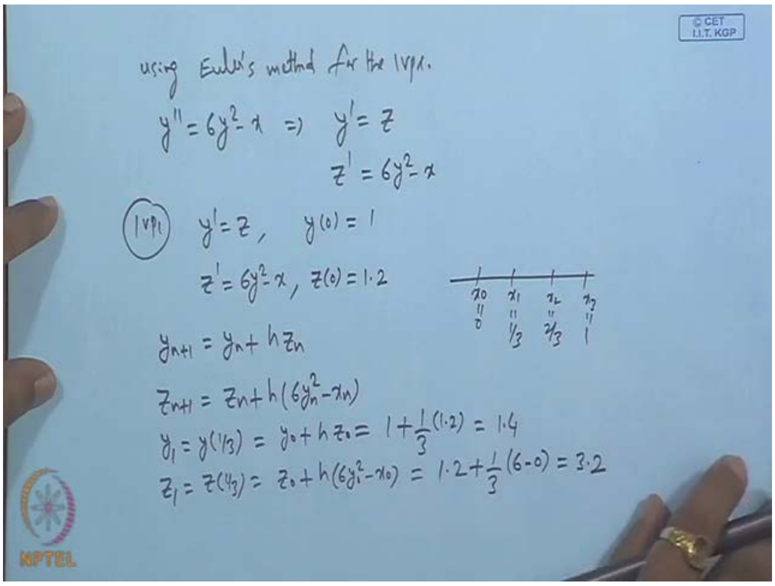

So, we define using Euler method for the IVP's, so we reduce to the system then IVP 1 y of 0 is one then y dashed which we have chosen is 1.2. Now, for this x 0 is 0. So, y n plus 1 is y n plus h y n prime that is z n z n plus 1 is z n plus h 6 y n square minus x n. Now, using this let us compute y 1 which is y of one third so y 0 so y 0 is 1 h and z 0 is 1.2. So, this will be 1.4 z 1 is z 0 plus h 6 y 0 square. So, z 0 is 1.2 plus one-third 6 y 0 square so that is 6 minus x 0 so this will be 3.2.

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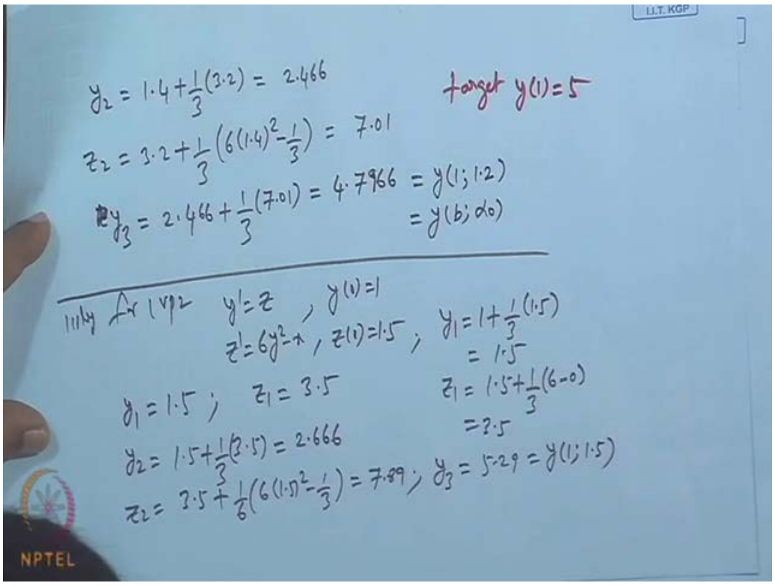

Then we proceed further y 2 this is y 1 plus h z 1, so this will be and z 2 z 1 h 6 y 1 square minus x 1. So, this have computed so z 2 then y 3, y 2, z 2, so we get y 3 so this is y at 1 using so this is y at b alpha 0. Similarly, for IVP 2 so we have y dash z, z dash y of 0 is 1 z of 0 is 1.5. So, we can obtain y one is same all the method y 1 is 1.5 z 1 is 3.5 then y 2 is you want me to write down.

So, y 1 is 1 plus 1 by 3, so this is 1.5 z 1 is z 0 plus h 6 y 0 1 minus. So, this will be then y 2 is y 1 plus h z 1 and z 2 z 2 if you compute z 1 1 by 6 6 y 1 square minus x 1. So, this is then y 3 is 5.29 and what is this y of 1 using 1.5, so either of them so this is 4.79 and what is our target so our target, target was so assuming either of them is a not close enough then we have to go for second method.

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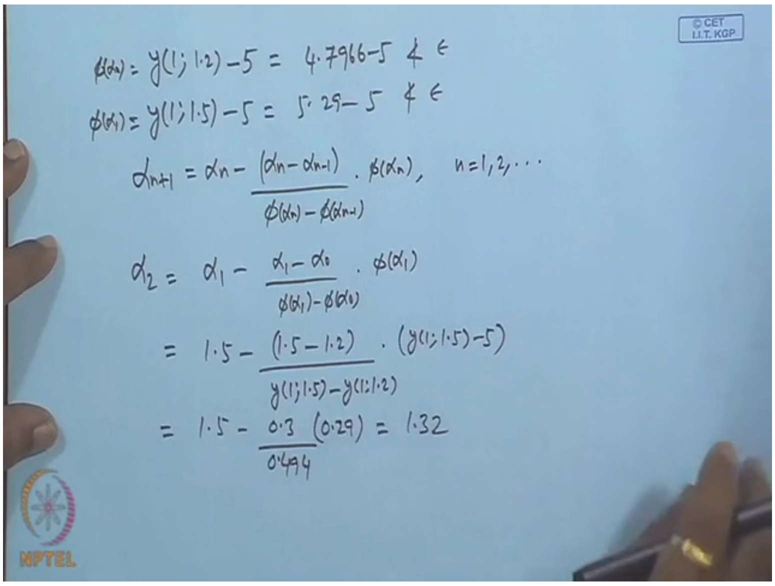

So y of 1 using 1.2 minus 5 minus 5, so this is 4.7966 minus 5 and y of 1 using 1.5 minus 5 this was 5.1, so assuming both are not less than epsilon, let us say. Now, let us define second method, so alpha 2 is alpha 1 minus, so these are essentially our 5 of alpha 0 and 5 of alpha. So, alpha 1, 1.5 then so this will be so gamma 2 get cancelled it is essentially y of 1 using 1.5 y of 1, 1.2 into 5 of alpha 1. So, this is y of 1 using 1.5 minus 5. So, this we have this 0.3 by this difference, so the difference of 1 at 1.5 this 5.29 difference this one and this difference we have. So, this is equals one point so this alpha 2. So having computed alpha two what we have to do?

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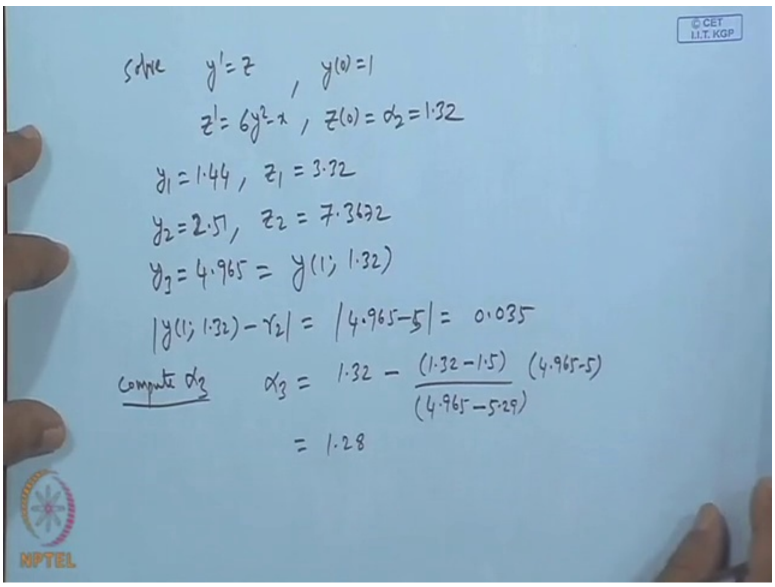

Solve z of 0 is alpha 2 which is 1.32 again Euler method we solve let us say on solving we get. So, I am not giving the details y 3, 4.965 and what is this y 3 y at 1 using 1.32 and is it close to five depends, so you can check so mod minus gamma 2. So, this is mod four point. So, this is so suppose you are not happy so this is not epsilon because so then what we do? Compute alpha 3, so alpha 3, alpha 2. So, this 1.28, so we are adjusting so then what we do?

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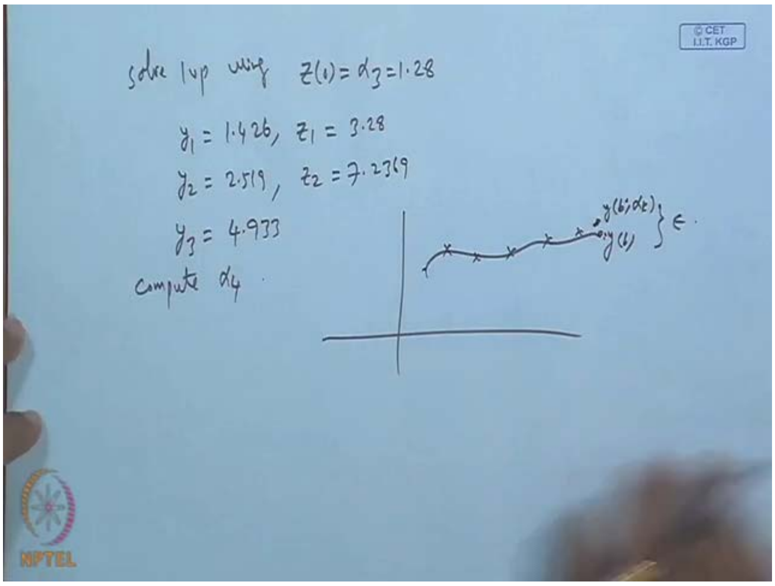

Solve IVP using z 0 equals 2 alpha 3 equals 1.28, so then we get let us say something like this then it is not close so then we have to again compute alpha 4. So, we have to continue the process like this, so when we are happy with that desired accuracy then we say we have reached the slope approximate slopes with which if we reach the target.

That means, the entire path has been so this is y of b say you have computed using initial value problems some error. So, this is computed with some alpha k. So, this distance is epsilon right so this is shooting method using Secant method, now we have discussed already while shooting method is applied all that we have to do is we have to refine our slope. Sometimes we are all shooting sometimes we are under shooting and all that, so can be refined slope using any other method so we have ah standard Newton Raphson method as well. So let us look into this using Newton Raphson method.

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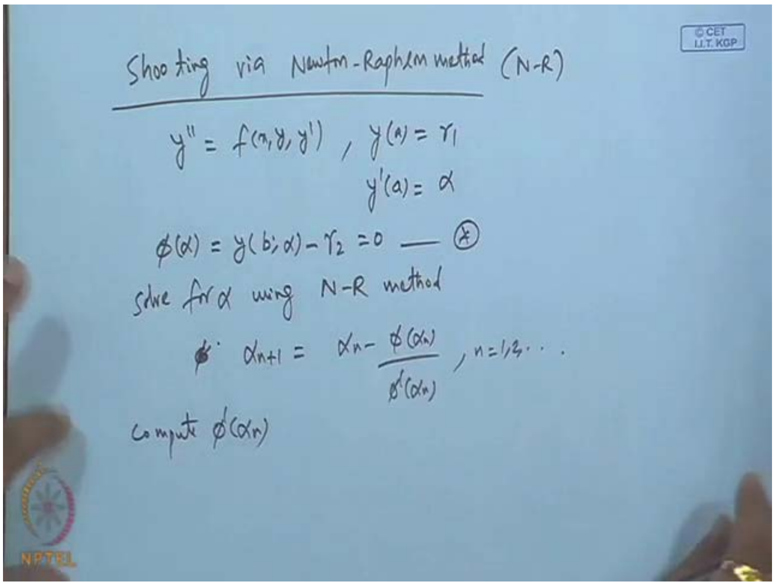

So, shooting and Newton Raphson so this is slightly tricky, so consider y double prime is so I have considered this type, complete non-linear so y dash of a is alpha so this is our alpha. Now, what is our phi of alpha, now the idea is solve for alpha using N-R method. Now, if you want to define Newton Raphson method, so we have to compute so the issue is how to compute pie dash of alpha n.

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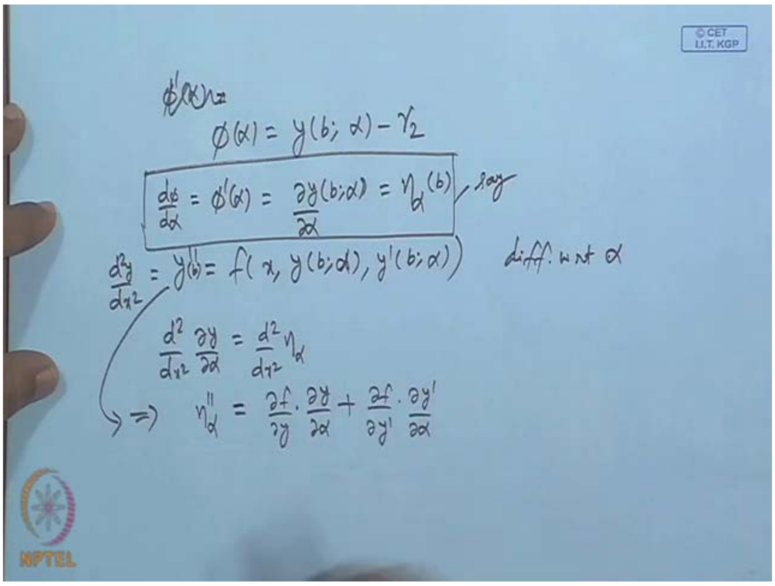

So let us phi dash of alpha equals see what was our phi let us start with phi of alpha is y of b alpha minus gamma two, then if I differentiate with alpha treating this as two variable function of this argument as well as the parameter. So, this let us call some eta of alpha b, say then we have y double is f of x, y of b s or alpha and y dash b alpha differentiate with respect to alpha both sides see this prime is x this is what?

Now, if I differentiate with respect to alpha, alpha goes in I can take of so this is eta alpha, so this is remember at b. So, we get eta alpha double, so prime is I am denoting, so this is taken there equals, now right hand we had differentiated with the alpha. So, this we use the chain rule x is independent dou f by dou y dou y by dou alpha plus dou f dou y prime dou y dou y.

So, please try to understand so this is right so this is the definition so dou y by dou alpha we are denoting by eta alpha, then we have this original given ode that written at b differentiating both sides with respect to alpha. So, I have shown the working, so this is the transmission so this reduces two of differentiate with respect to alpha, left had side become this and right hand becomes using chain rule this.

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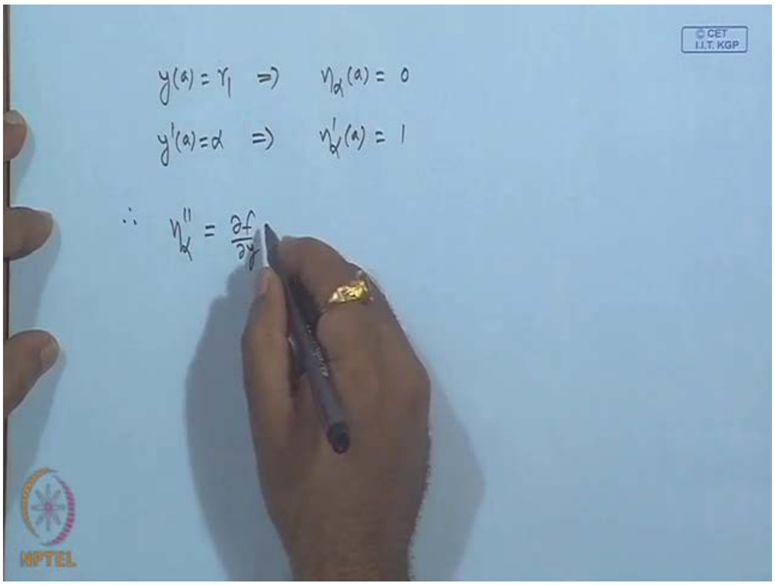

Then we have y of a gamma 1 if you differentiate with alpha and right hand side is 0 and we have initial condition, if you differentiate with the alpha this will be this is not derivative just to know that it depends on alpha and right hand side is one. Therefore, what did we achieve, we have converted and one more, this is what? This is eta alpha and what about this prime can be given to eta right, how see for example.

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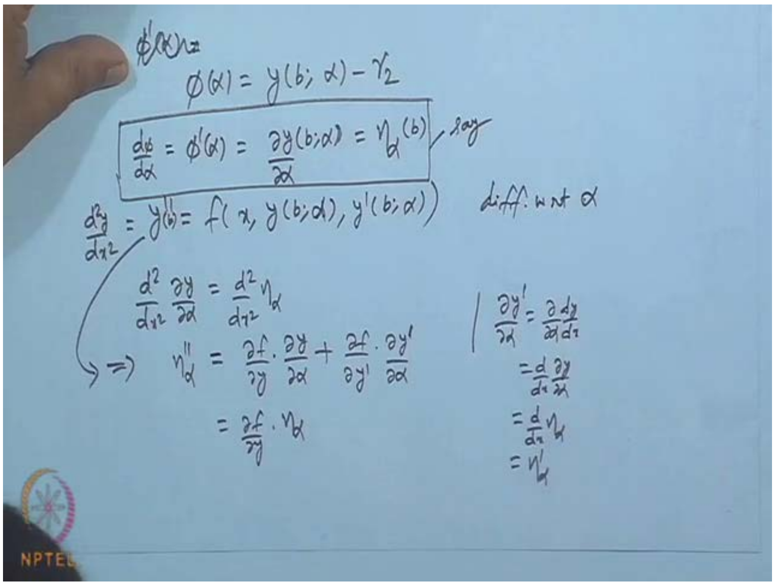

We are looking for so this will be d by d x of eta alpha so that will be prime.

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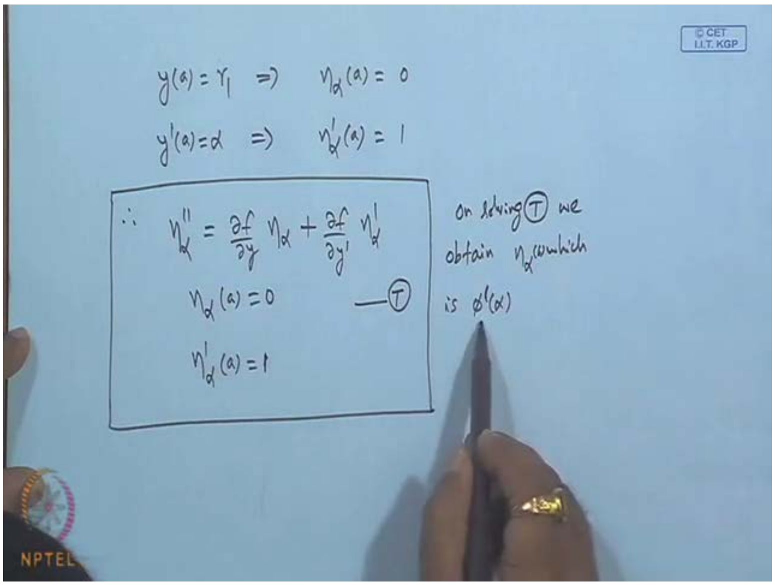

So, with this notation this becomes eta alpha and eta alpha a is 0 1. So, what did we achieve in order to find phi of alpha we have obtained this. So, essentially phi dash of alpha if we solve see we are looking for phi dash of alpha in order to use in our Newton Raphson method now what did we obtain? So this is say some on solving T, we obtain eta alpha which is phi dash that of alpha. Of course, b so this gives a big tool to compute phi dash that of alpha, so let us see for example.

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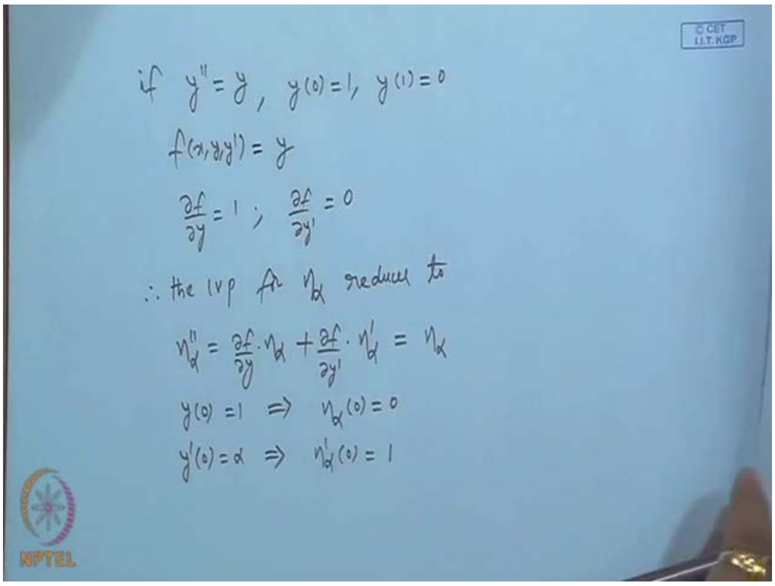

If y double is y simplest case we are taking say y 0 is 1 and y of 1 is 0 is our boundary value problem. Now, what is our f our f was just y therefore, 1 is 0 therefore, the IVP for eta the IVP for eta alpha reduces to what was our IVP? So, this reduces to dou of dou y is 1 and this is 0 and boundary conditions so we have to find. So, this implies and y dash of 0 is alpha this implies 1.

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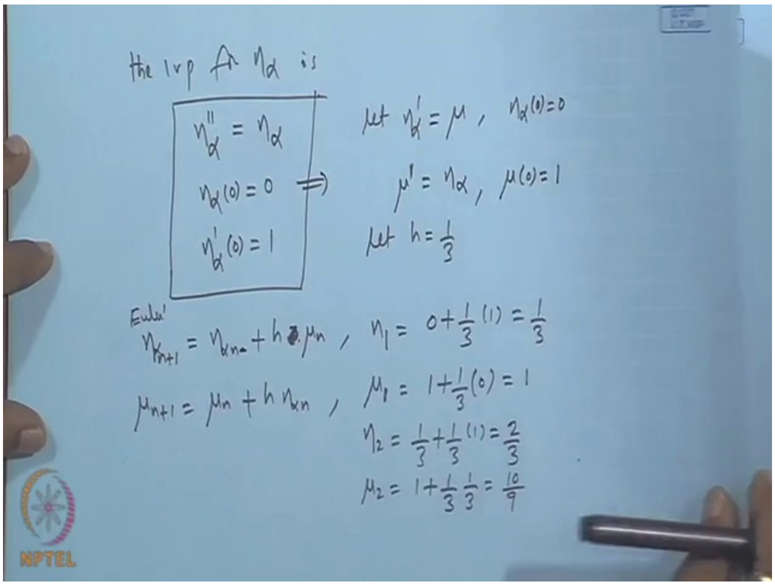

So, accordingly the IVP is eta alpha prime is eta alpha 0, now how do we solve this again system let eta alpha is some mu then mu prime is eta alpha. Accordingly then mu of so this is the system so let h is 1 by 3. So, then we have to define Euler method of course, equals eta alpha n plus h z n sorry h mu n then mu n plus 1 mu n plus h eta alpha n. So, from this let us solve eta 1, I am dropping alpha because too many suffixes so eta 1 is eta 0 plus h mu 0 and mu 1 is mu 0 plus h eta 1 eta 1. So, that is eta mu 1, mu 1 is mu 0, plus mu 0 plus h eta 0, so this is 0 so this is one. Similarly, eta 2 is eta 1 plus h to mu 1, so this will be and mu 2 mu 1 h eta 1, so this will be see essentially we are solving this using Euler method.

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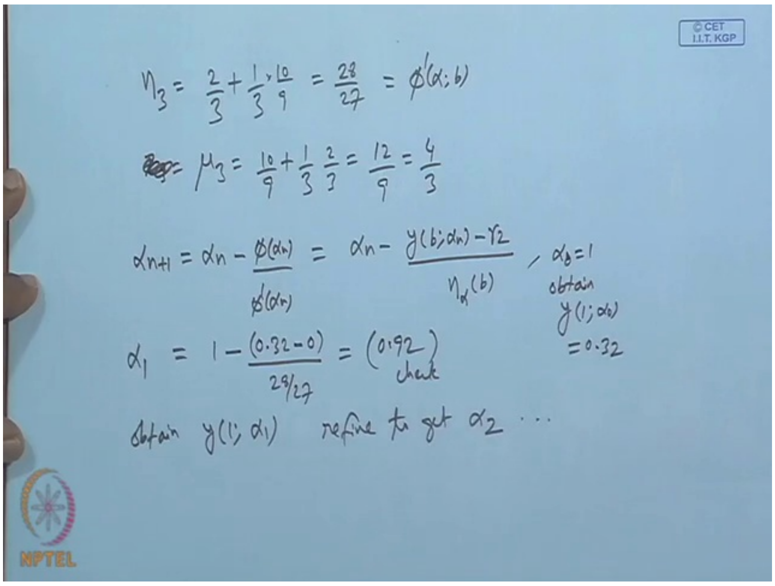

So, ultimately we need a eta 3 so eta 2 plus h mu 2, so this will be 9, so this is our value mu 2 is so this will be so we have obtained eta 3. So, this is nothing but phi dash of alpha of course, with this b right, therefore we can obtain phi of alpha n. So, this is alpha n minus y of b at using alpha n minus gamma 2 and this is eta, so this is alpha 1. So, let us choose say alpha 0 is 1 and this we have to solve with alpha 0 obtained y at 1 using alpha 0. Say this is point suppose same problem we have to solve so if we solve, let us say this is the value. So, this the value we have obtained minus gamma 2 so gamma 2 was the given gamma 2 was 0 and eta alpha just now we obtained. So, this is so we get defined value.

So, this is our alpha one then again we have to obtain, so let us say this is this is some one point right, so we get some value let us say this is some 0.92 check. So, then obtain y at 1 using alpha 1 again refine to get alpha 2. So, we continue until we get the ah desired accuracy, so shooting method the idea is same either, we use secant method or we use Newton Rahpson method, the idea remain the same. Time and again we have to solve several initial value problems for each mu slope we solve the mu initial value problem with that slope and then try to obtain the solution. And check how far we are from the target then if you are not satisfied refine your new slope to get a new slope using secant method. Only difference is in case of Newton Raphson to solve the phi dash of alpha again we have to solve a system of equations for finding phi dash of alpha.

So, that means one set of IVP for the original problem and another set of ivp is for eta, which is our slope five dash of alpha. So, this is a slightly complicated, otherwise the underline principle remain the same. So, with this we have reviewed more or less different methods for two point boundary value problems. So, in the next class we will discuss further more topics related to 2.1 value problems and do some problems bye.