Introduction to Nonlinear Dynamics
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Module -03
Lecture-07
1-Dimension Flows, Flow on the line, Lecture 5

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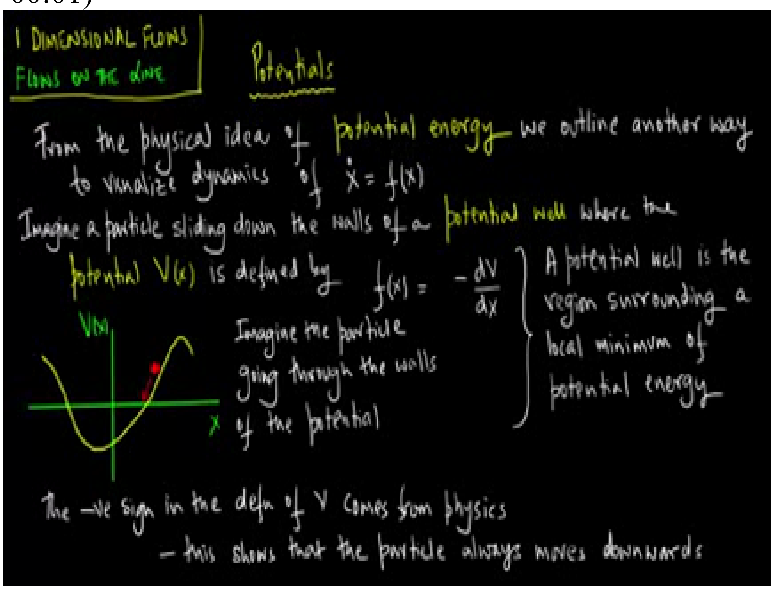

This is a short lecture on potentials. So from the physical idea of potential energy, we outline another way to actually visualise the dynamics of $\dot{x} = f(x)$. So you imagine a particle sliding down the walls of a potential well, where the potential V(x) is defined by $f(x) = -\frac{dv}{dx}$. A potential well is the region that is surrounding a local minimum of potential energy. So let us make a simple minded plot of V(x) versus x. That is your potential well.

We highlight the particle and the direction in which it is moving. So imagine the particle actually moving through the walls of the potential, the negative sign in the definition of V actually comes from physics, essentially what this shows is that the particle always moves downwards.

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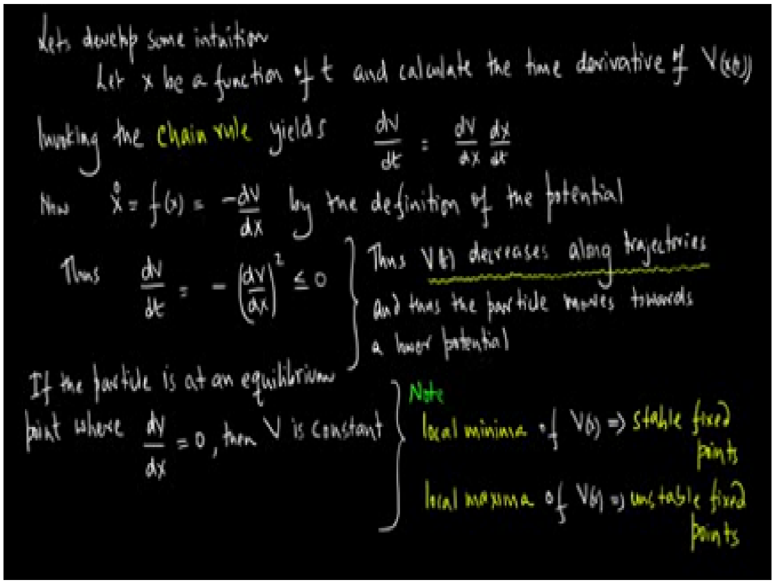

Let us go ahead and develop some intuition for ourselves, let x be the function of t and let us calculate the time derivative of V is the function of x(t). So, invoking the good old chain rule from calculus yields $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$. So now $\dot{x} = f(x) = -\frac{dv}{dx}$ and that is simply by the definition of the potential. Thus, $\frac{dv}{dt} = \left(\frac{dv}{dx}\right)^2$, which should be $\leq$ 0.

So V(t) decreases along the trajectories, that's worth highlighting and thus the particle moves towards a lower potential. Now if the particle is at an equilibrium when $\frac{dv}{dx} = 0$ and so V is simply a constant. Now note that the local minima of V(x) gives us stable fixed points and the local maxima of V(x) gives us unstable fixed points.

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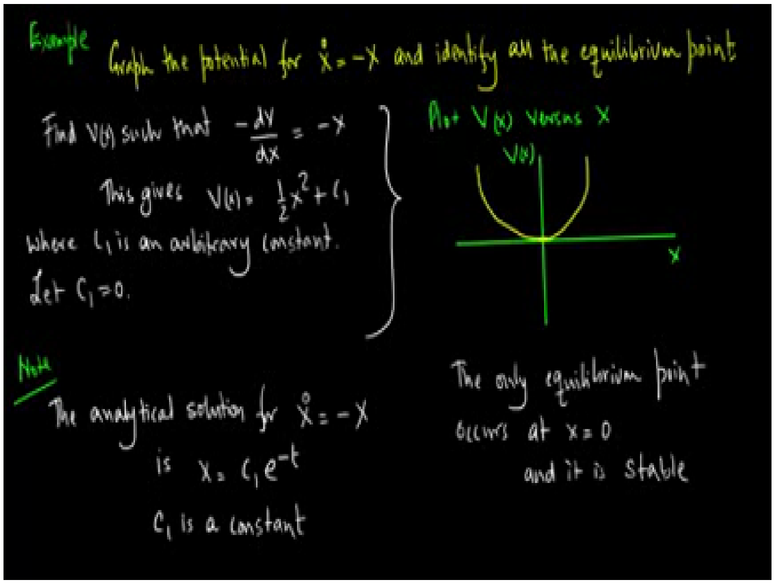

Let us consider an example, graph the potential for $\dot{x} = -x$ and identify all the equilibrium points.

So, we need to find V(x) such that $-\frac{dv}{dx} = -x$. This gives us $V(x) = \frac{1}{2}x^2 + c_1$, where C1 is just an arbitrary constant. So for now we let C1 be 0, now let us plot V(x) versus x, this plot of V(x) versus x is rather simple minded curve, which we can easily do by hand and there we go that is what the curve looks like.

The only equilibrium point occurs at x = 0 and it is stable. The analytic solution for $\dot{x} = -x$ is just $x = c_1 e^{-t}$, where C1 is a constant.

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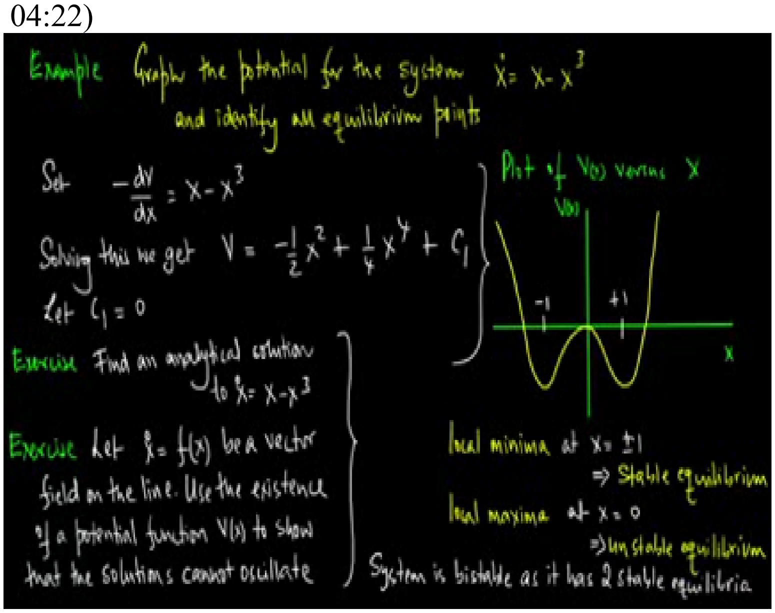

Let us consider another example, graph the potential for the system $\dot{x} = x - x^3$ and identify all equilibrium points. So, we set $-\frac{dv}{dx} = x - x^3$ and solving this we get $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + c_1$. Let C1 = 0. Now let us make the plot of V(x) versus x, so the plot of V(x) versus x is little bit more involved highlight the local minima +1 and -1.

So, the local minima is at x = ∓ 1 which implies stable equilibrium and the local maxima is at x = 0, which implies unstable equilibrium. The system is bistable as it has 2 stable equilibria. So here is an exercise, can you find an analytical solution to $\dot{x} = x - x^3$. So, let us consider another exercise, let $\dot{x} = f(x)$ be a vector field on the line and use the existence of a potential function V(x) to show that the solutions actually cannot oscillate.

So, this second exercise is actually closely related to the lecture where we talked about the impossibility of oscillations of $\dot{x} = f(x)$. But here what I am saying is that can you use the existence of a potential function V(x) to actually show that solutions of $\dot{x} = f(x)$ cannot oscillate.

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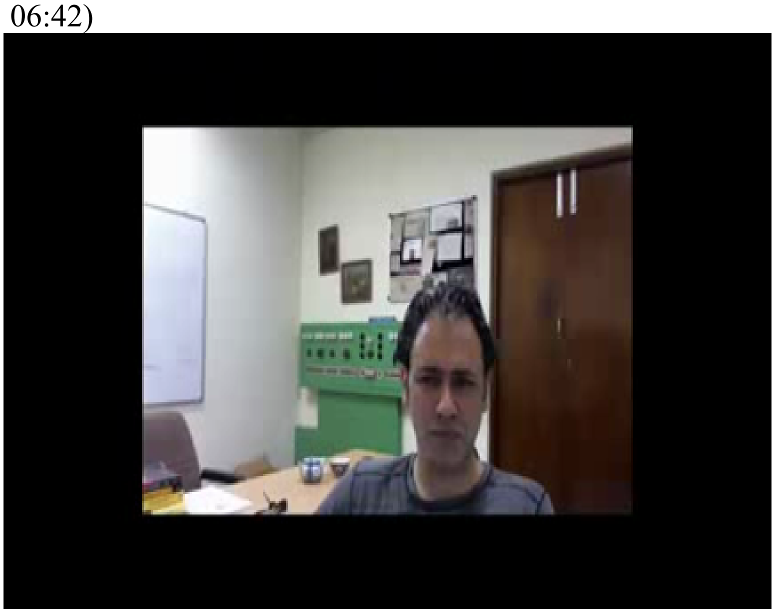

Now, this was a very short lecture, the intent of the lecture was to introduce you to the notion of potentials and to highlight their ability to analyse equations of the form $\dot{x} = f(x)$. Now you look at the definition of a potential. So let us assume that we have potential function V(x) which is defined as $f(x) = -\frac{dv}{dx}$, then evaluating that, relationship allowed us to say something about the original nonlinear system $\dot{x} = f(x)$.

We offered one of two examples, but we left you with an interesting exercise that I suggest that you actually try which was roughly as follows, now can we actually use the notion of a potential as applied to an equation of the form $\dot{x} = f(x)$ and prove using this notion that the solutions of $\dot{x} = f(x)$ will actually not oscillate. They will actually not oscillate, this is something that we are talked about earlier in the lectures in terms of impossibility of oscillations, but now can you use this notion of potential to make exactly the same point again.