Introduction to Nonlinear Dynamics
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Module -02
Lecture-05
1-Dimension Flows, Flow on the line, Lecture 5

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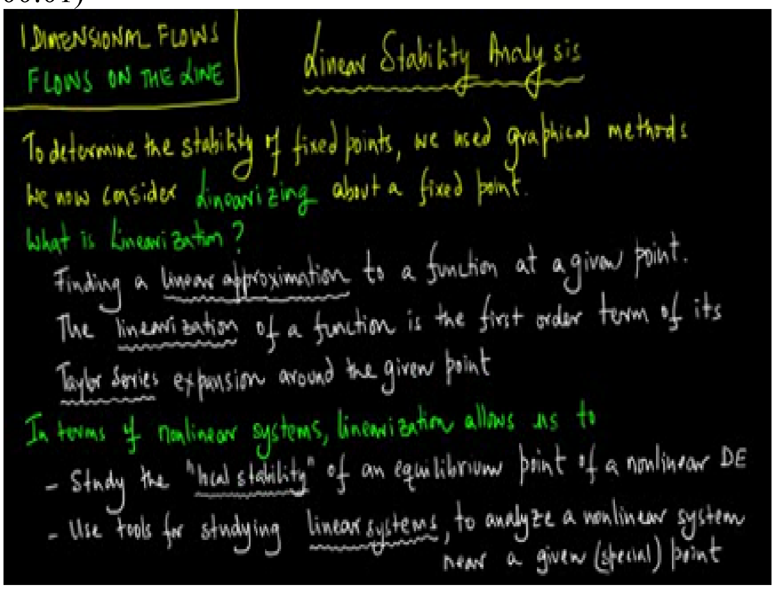

Now in this lecture, we will focus on linear stability analysis to determine the stability of fixed points. We had earlier used graphical methods. We now consider linearizing about a particular fixed point, what really is linearization? Well it is finding a linear approximation to a function at a given point. So, the linearization of a function is the first order term of its Taylor series expansion around the given point.

In terms of nonlinear systems linearization allows us to study the local stability of an equilibrium point of a nonlinear differential equations. And it allows us to use tools for studying linear systems to analyse a nonlinear system near a given special point.

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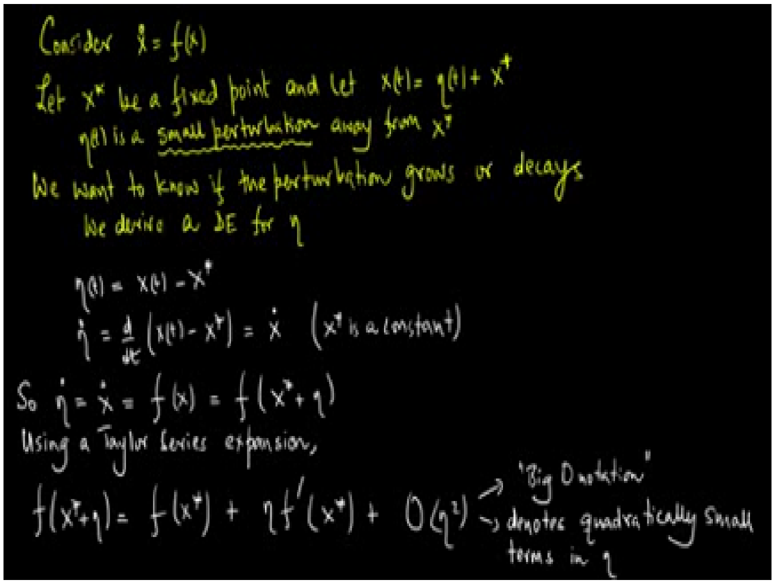

Now consider $\dot{x} = f(x)$ let x* be a fixed point and let $x(t) = \eta(t) + x^*$ where $\eta(t)$ is a small perturbation away from x*. We really want to know if the perturbation actually grows or decays.

So we go ahead and derive a differential equation for η. So, $\eta(t) = x(t) - x^*$, so $\dot{\eta} = \frac{d}{dt}(x(t) - x^*) = \dot{x}$ as x* is simply a constant.

So $\dot{\eta} = \dot{x} = f(x) = f(x^* + \eta)$ and so now we go ahead and using a Taylor series expansion we get $f(x^* + \eta) = f(x^*) + \eta f'(x^*) + O(\eta^2)$. Note that this is the big "O" notation where this term actually denotes quadratically small terms in eta.

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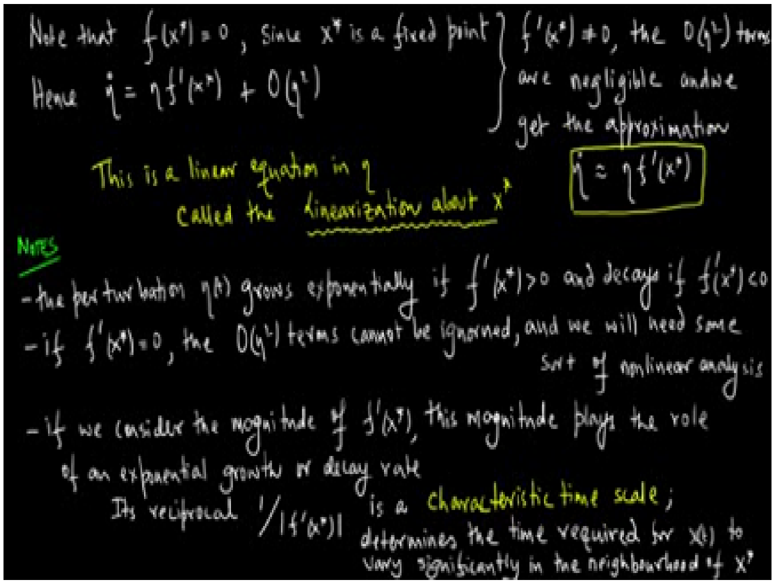

Note that f(x*) is equal to 0 since x* is a fixed point. Hence $\dot{\eta} = \eta f'(x^*) + O(\eta^2)$. If f prime of x* is not equal to 0 the order η^2 terms are in fact negligible and we get the following approximation $\dot{\eta} = \eta f'(x^*)$. Now this is a linear equation in eta which is called the linearization about x*. Now here are some notes the perturbation etas of t grows exponentially, if f prime of x* > 0 and decay if f prime of x star < 0. If f prime of x* = 0 the order η^2 terms cannot be ignored and we will need some sort of nonlinear analysis for the equation. Now if you consider the magnitude of f prime of x* then this magnitude plays a role of an exponential growth or decay rate. It is reciprocal one upon f prime of x* is a characteristic time scale. Now this determines the time required for x of t to vary significantly in the neighbourhood of the fixed point x*.

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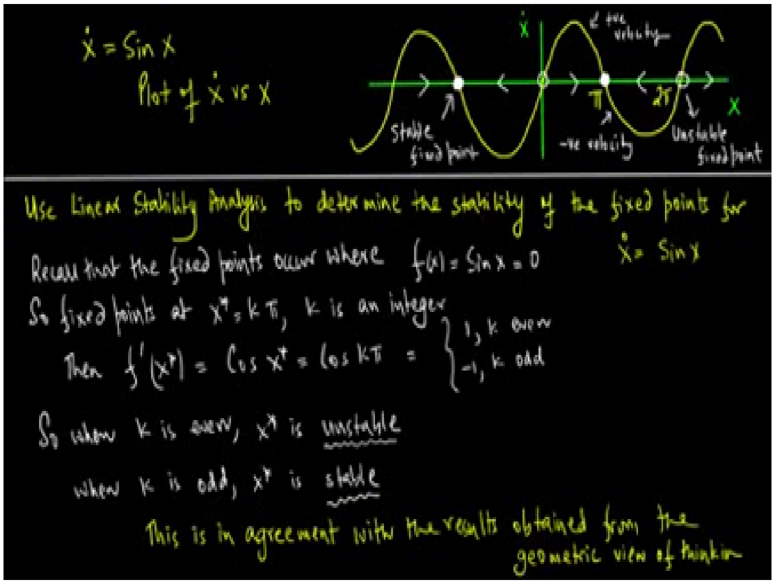

Now let us consider this equation $\dot{x} = \text{Sin}(x)$. Now recall that from the geometric view of thinking essentially, what we said was that when we have equation of form $\dot{x} = f(x)$ then what we first do is just plot x dot versus x. So for this particular equation where f(x) is sine x. So what we see here is a plot of x dot versus x where using the geometric view of thinking that we have outlined earlier we were able to classify the stability of the fixed points noting that you were able to do so without any formal analysis.

And so now we go ahead and use the linear stability analysis to determine the stability of the fixed points for the equation x dot is equal to sine x. Recall that the fixed points occur where $f(x) = \text{Sin}(x) = 0$. So the fixed points are at x* = Kπ where K is an integer then $f'(x^*) = \text{Cos}(x^*) = \text{Cos}(K\pi)$, which is equal to 1 if K is even and -1 if K is odd.

So when K is even x* is unstable and when K is odd then x* is stable, so this is actually in complete agreement with the results that we obtained from the geometric view of thinking. Now, note that with geometric view of thinking, we had no analytical basis really to establish the stability of the fixed points and it was purely geometric. But now we actually have an analytical and algebraic technique to actually establish the stability or the instability of the fixed points.

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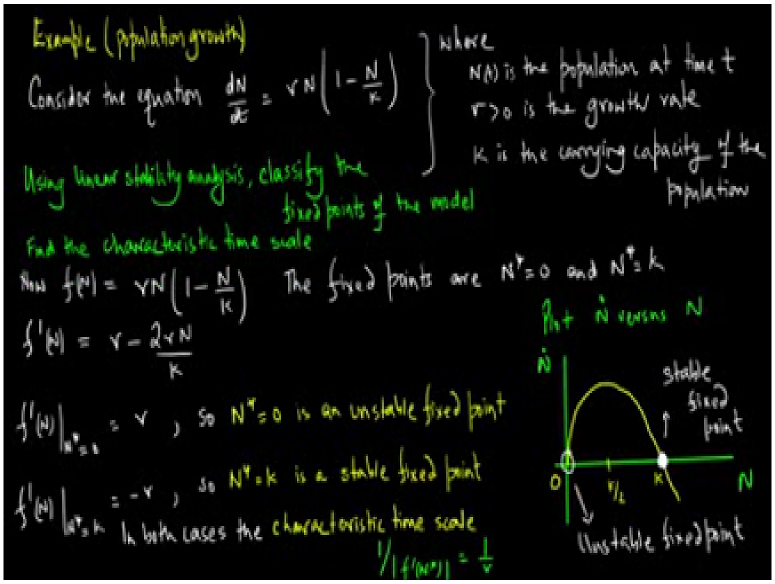

Now let us consider examples that arises in population growth consider the equation $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$, where N of t is the population at time t, r > 0 is the growth rate and K is carrying capacity of the population. Our objective is that using linear stability analysis, we wish to classify the fixed points of the model and we also wish to find characteristic time scale of the system. Now $f(N) = rN\left(1 - \frac{N}{K}\right)$. So the fixed points are N* = 0 and N* = K evaluating $f'(N) = r - \frac{2rN}{K}$, so f prime of N at N* is equal to 0 is r. So N* = 0 is actually an unstable fixed point. And f prime of N evaluated at N* is equal to K is minus r, so N* = K turns out to be a stable fixed point. In both of the above cases the characteristic time scale turns out be the same, so $\left|\frac{1}{f'(N^*)}\right| = \frac{1}{r}$ for both the fixed points. Now let us go ahead and plot N dot versus N by now we are quite familiar with making such plots. So that is the plot for N dot versus N, 0 and K the two fixed points have been highlighted and K represents the stable fixed point and 0 represents the unstable fixed point.

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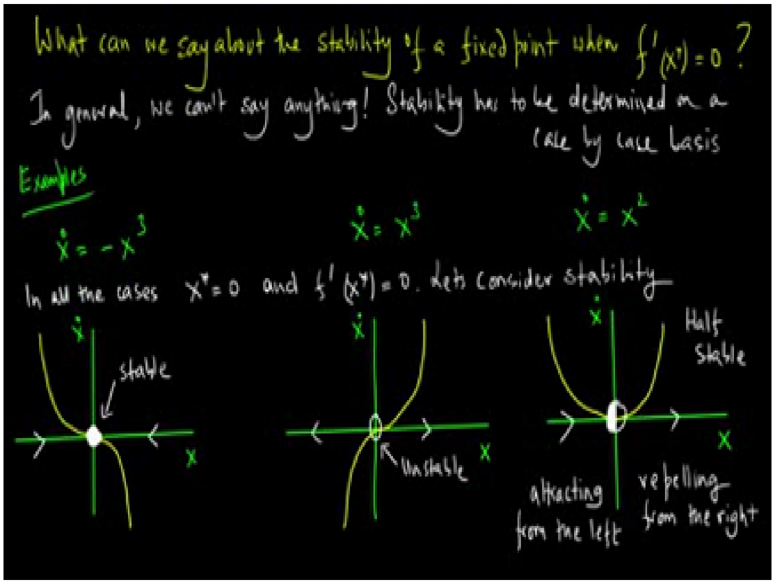

So what can we say about the stability of a fixed point when $f'(x^*) = 0?$. As a matter of fact in general we can't say anything stability has to be worked out and determined on a case by case basis. Now let us go ahead and consider some examples, $\dot{x} = -x^3$, $\dot{x} = x^3$ and $\dot{x} = x^2$ in all the cases x* = 0 and $f'(x^*) = 0$. So now let us go ahead and consider stability in each of these cases.

So we plot x dot versus x and what we find is that we have an attracting stable fixed point and similarly plotting x dot versus x in this particular case we find that we have an unstable fixed point. So the last case actually presents us with a rather interesting situation. We go ahead and plot x dot versus x and that is the plot. So we find that the fixed point is attracting from the left and it turns out be repelling from the right.

So we get the situation where the fixed point turns out to be stable on the one side but unstable on the other side. So such a situation is referred to as a half stable fixed point attracting from one end and repelling from the other.

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Ok so let us do a quick recap of this lecture. This lecture was essentially about linear stability analysis. We are still dealing with the equations of form the x dot is equal to f(x) and essentially the idea was that you identify the equilibrium point. You take a Taylor series expansion of the original nonlinear system around the equilibrium point. You retain the linear terms and all higher order terms are actually thrown out. So you are essentially left with linearized version of the original nonlinear system.

And now you can go ahead and apply all the tools and methods that you learned for linear system analysis to this particular equation. So that is a big advantage. The disadvantage is that you actually thrown out all the nonlinear terms out ok. So if you thrown all the nonlinear terms out all the fun all the interesting dynamics are essentially been thrown out, it is a good start, but it is only a start, so then what we did was we took a few examples x dot is equal to sine x. We took another example, where which was motivated from a population dynamics.

And then we essentially showed that if you conduct a geometric reasoning around nonlinear system which is by plotting x dot versus x or if you did a linear stability analysis then you got essentially the same results. Except that we left you with some food for fork with last example. The last example was rather interesting one, now essentially what you actually had was that you had was fixed point, which it turned out be attracting from one end but repelling from the other end.

So the fixed point could not really be classified has an stable fixed point or an unstable fixed point and the terminology we use was a half stable fixed point right and I think we were just we leave this lecture with that particular food for fork with you that you could also end up you know fixed point would essentially not be just stable or unstable. But you can an actually have this sort of hybrid cases which arise, which is half stable and half unstable.