Introduction to Nonlinear Dynamics
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Module -02
Lecture - 03
1-Dimension Flows, Flow on the line, lecture 1

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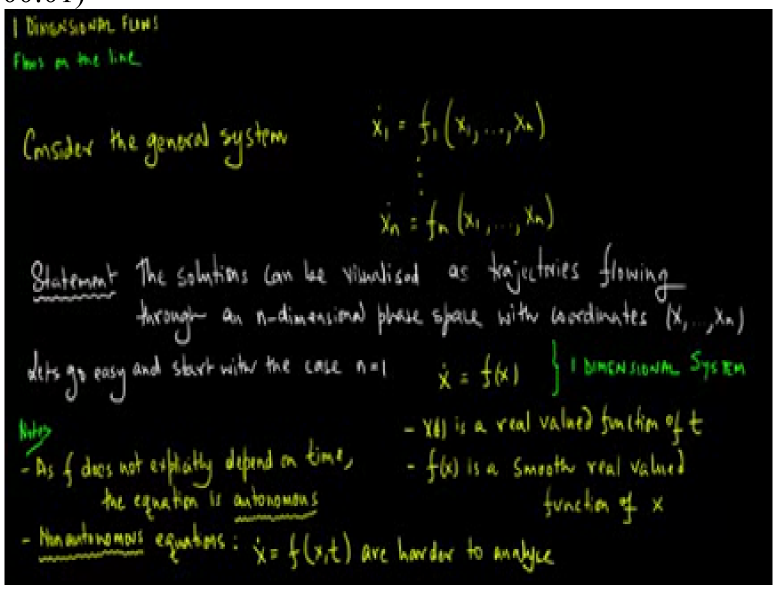

We start with one dimensional flows and we will focus on flows on the line. Consider the following general nonlinear system, $\dot{x}_1 = f_1(x_1, ...x_n)$ all the way up to $\dot{x}_n = f_n(x_1, ...x_n)$. Now let us make a statement about the above system, the solutions can be visualised as trajectories flowing through an n dimensional phase space with coordinates X1 all way to Xn.

Now let us actually go bit easy and start with the case n is equal to one, that is $\dot{x} = f(x)$ where x(t) is the real valued function of t and f (x) is a smooth real valued function of x. So this is an example of a one dimensional system. Now here are some notes as the function f does not explicitly depend on time the resulting equation is autonomous. Non-autonomous equations are the equations of the form $\dot{x} = f(x,t)$ and are in general much harder to analysis.

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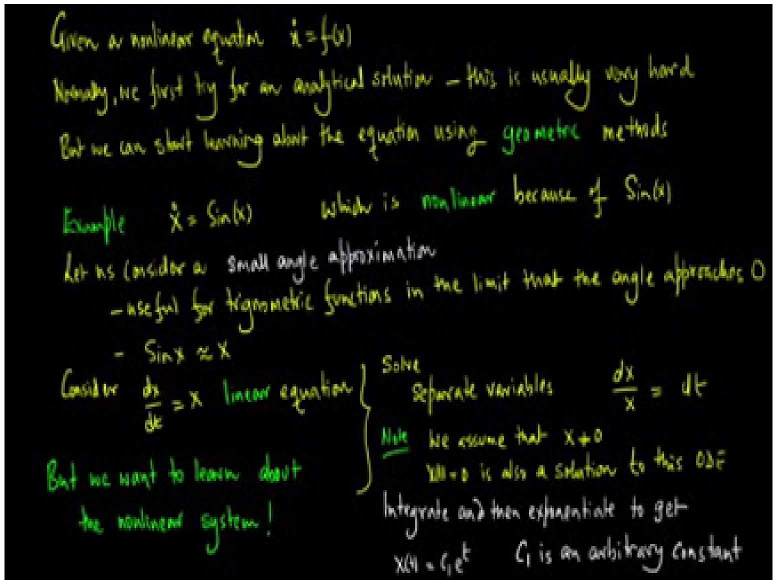

Now given a nonlinear equation $\dot{x} = f(x)$ normally we first try and look for an explicit analytical solution. This is usually very hard, but we can start learning about the equation using geometric methods. Here is an example, $\dot{x} = \mathrm{Sin}(x)$ which is nonlinear because of the sine x term. Now let us actually first consider a small angle approximation, which is useful for trigonometric functions in the limit that the angle approaches zero so sine x is approximated as x.

So now consider $\frac{dx}{dt} = x$ which is a linear equation. So to solve it we separate variables to get $\frac{dx}{dt} = dt$. Note that we are assuming that $x \neq 0$. However the $x(t) = 0$ is also a solution to this ordinary differential equation. So we integrate and then exponentiate to find a solution $x(t) = c_1 e^t$, where C1is arbitrary constant. But what we really wanted to do was to learn about the original nonlinear system.

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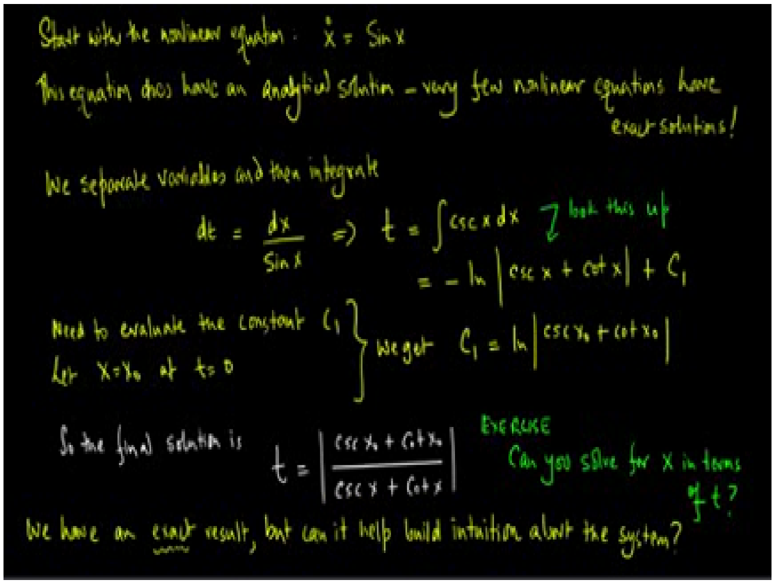

Now let us actually start with the nonlinear equation $\dot{x} = \mathrm{Sin}(x)$. Now this equation actually does have an analytical solution, but very few nonlinear equations actually admit exact solutions. So to solve the equation we separate the variables and then integrate, so $dt = \frac{dx}{\mathrm{Sin}(x)}$, which gives us $t = \int \csc(x)dx$ evaluating this integral we get $t = \int \csc(x)dx = -\ln(\csc(x) + \cot(x)) + c_1$. We suggest that you actually look this integral up, we need to evaluate the constant C1, so we let $x = x_0$ at $t = 0$ and we get $c_1 = \ln(\csc(x_0) + \cot(x_0))$. So the final solution turns out to be $t = \frac{\ln(\csc(x_0) + \cot(x_0))}{\ln(\csc(x) + \cot(x))}$. So here is an exercise. Can you solve for X in terms of t you should do this on your own. Now we have an exact result but can it actually help to build intuition about the original nonlinear system.

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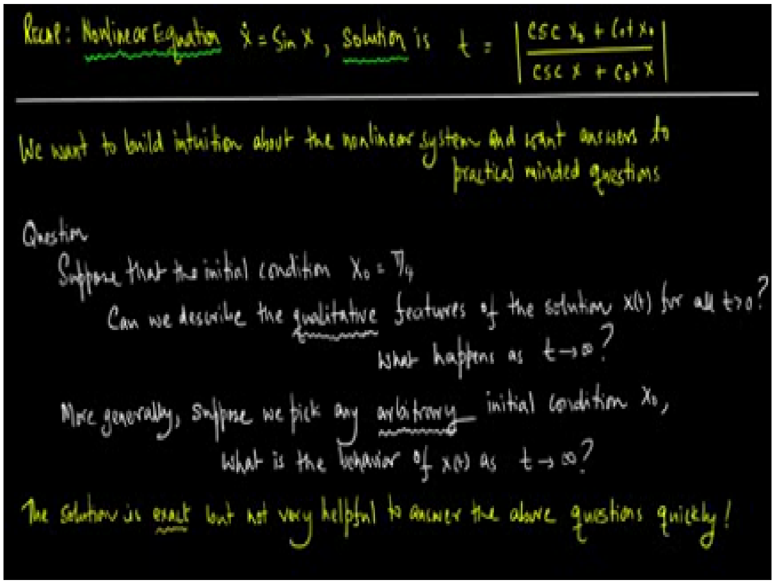

Now, let us just recap what we have, we have a nonlinear equation $\dot{x} = \text{Sin}(x)$ and using some analytic methods we were able to obtain an exact analytical solution to this particular equation. Now what we really want is to build intuition about the nonlinear system and be able to answer practical minded questions.

For example suppose that the initial conditions is $x_0 = \pi / 4$ then can we describe the qualitative features of the solutions X(t) for all t greater than zero. What happen as t tends infinity more generally, suppose we pick any arbitrary initial condition X of 0, then what is the behaviour of X of t as t tends to infinity. Now the explicit solution is exact, but not extremely helpful to answer the above questions quickly.

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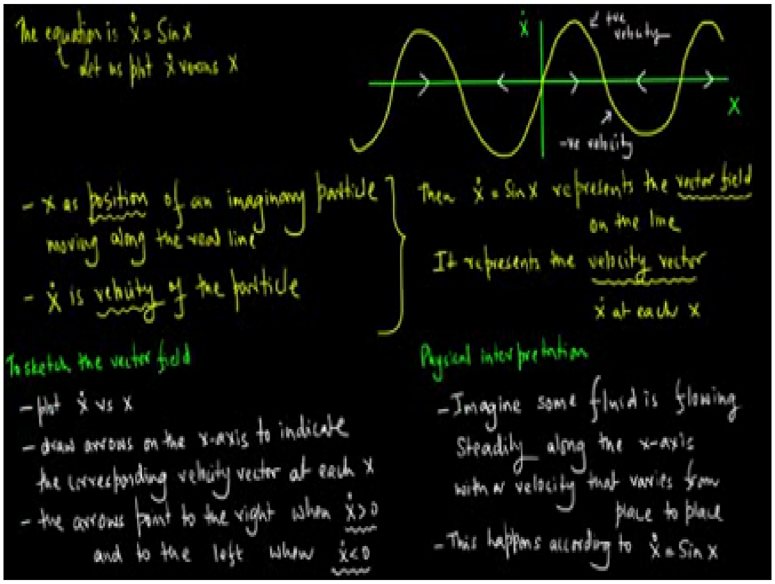

Now the nonlinear equation at hand is $\dot{x} = \mathrm{Sin}(x)$ so let us actually plot x dot versus x. Now here is slightly simple minded plot of x dot versus x. Now think of X as the position of an imaginary particle, that it is moving along the real line then x dot is the velocity of the particle then $\dot{x} = \mathrm{Sin}(x)$ represents the vector field on the line. Now, essentially it represents the velocity vector x dot at each x.

Now to plot the vector field we do the following, plot x dot versus x. Draw arrows on the x axis to indicate the corresponding velocity vector at each x, the arrows should point to the right when x dot is greater than zero and they should point to the left when x dot is less than zero. Now, please pay very close attention to the plot of x dot versus x.

We highlight the region of positive velocity where the arrows point to the right and note the region of negative velocity when the arrows point to the left. Now let us actually offer some physical interpretation, imagine that we have some fluid that is flowing steadily along the x axis with a velocity that actually varies from place to place then this is essentially just happening according to $\dot{x} = \mathrm{Sin}(x)$.

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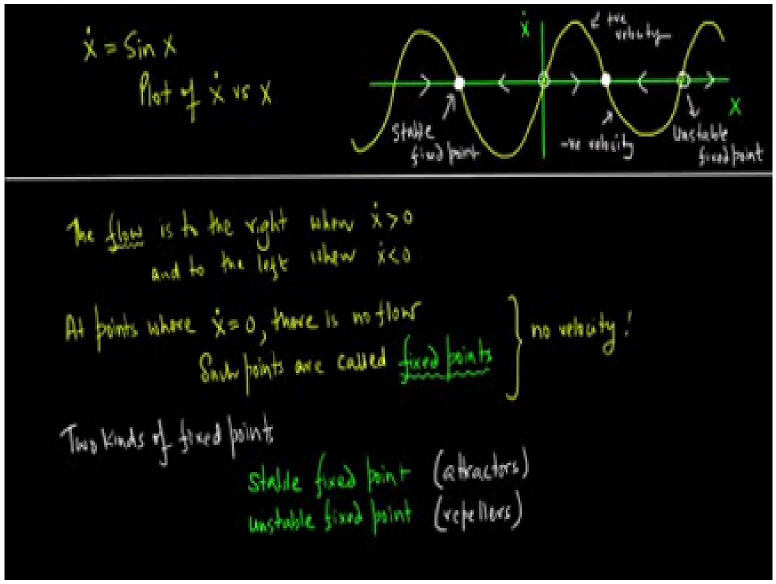

Now the flow is to the right when x dot is greater than zero and the flow is to the left when x dot is less than zero at points where x dot is equal to zero there is actually no flow and such points are called fixed points these are points where there is no velocity. Now there are two kinds of fixed points, stable fixed points also referred to as attractors and unstable fixed points referred to as repellers.

Now look at the diagram the closed circles are the stable fixed points and the open circles are the unstable fixed points. Note that on the stable fixed points the flow is getting attracted towards them and on the unstable fixed points the flow is getting repelled from them.

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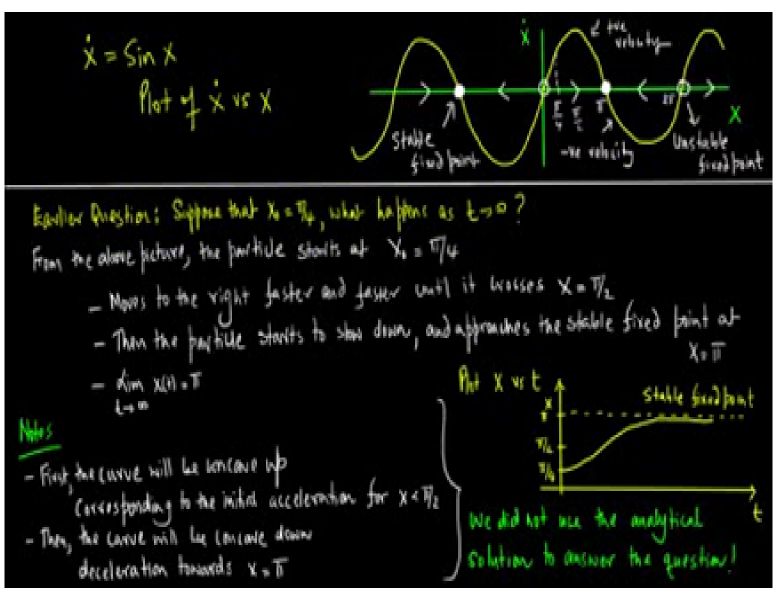

What we really want to do now is get some additional insight into this nonlinear system that we have. Now recall the earlier question suppose that $x_0 = \pi/4$ then what really happens as t tends to infinity. Now from the above picture the particle starts at $x_0 = \pi/4$ and then moves to the right faster and faster until it crosses $x_0 = \pi/2$ then the particle slowly starts to slowdown and approaches the stable fixed point at $x_0 = \pi$.

So the limit as t tends to infinity $x(t) = \pi$ if the particle actually as started at $\pi/4$. Here are some notes, first the curve will be concave up corresponding to the initial acceleration for X less than pi on two. Then the curve will be concave down highlighting deceleration towards $x = \pi$. Now let us go ahead and plot X versus t. We first highlight X is equal to pi which is our stable fixed point, highlight pi on two and pi on four which was our initial condition and that is the curve of X versus t. Observe that we did not use the analytical solution to actually answer the above question.

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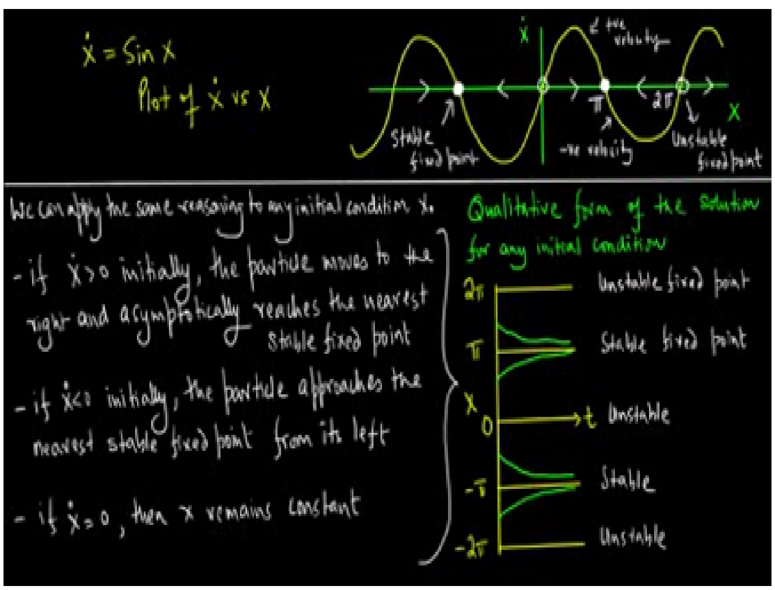

We can apply the same reasoning to any initial conditions X0. If x dot is greater than zero initially the particle moves to the right and asymptotically reaches the nearest stable fixed point. If x dot is less than zero initially, the particle approaches the nearest stable fixed point from its left and if x dot is equal to zero then x remains constant.

The qualitative form of the solution for any initial condition can actually be plotted using the rules that we have identified on the left. Here is the snapshot of what the solution should look like. Note that the trajectories are converging towards x is equal to pi and x is equal to minus pi and the reason is because these are the stable fixed points.

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Ok so now let us just rap up this lecture with some concluding remark. We started our study of nonlinear system with equations of the form x dot is equal to f of x. These were one dimensional flows and in particular, we were rather keen to understand flows on the line. Now when you are given a nonlinear equation the first thing you normally try, is to try and get an explicit analytical solution. Now, the first thing to remember is that explicit analytical solutions are usually very, very difficult to get for nonlinear system; ok.

So we have a particular example we said let x dot is equal to sine x. Now in this particular case we were able to get an explicit analytic solution we separated the variables and we integrated and the integral worked out explicitly. And in this case, we were able get an explicit analytic solution. However, the form of the solution was not very easy to understand it was not very easy to absorb its certainly was not easy to develop intuition about the form of the solutions.

So what we did was we went on to look at some geometric reasoning. Now essentially, all we did was we plotted x dot versus x, for this particular equation x dot is equal to sine x. Now as soon as we did that we found that there were interesting cases that showed up when x dot was greater than zero, when x dot was less than zero and when x dot was actually equal to zero, we encountered fixed points for the first time. Now such fixed point we found would either be stable or they could be unstable.

And now with this geometric form, we were able to ask and start answering questions of the form that if you chose a particular initial condition, what would be the solutions look like as t tends to infinity. So in particular, we just picked X0 is equal to pi on four as initial condition and ask what happens as time carried on. So the lesson learnt from there was that geometric reasoning can certainly complement analytical solutions in the case that we can find analytical solutions and in the case, that we actually can't find analytical solutions it is a good way to actually start yah. And but we have to be very careful that geometric reasoning may not be able to give all the answers we want. For example, if we were asked after a particular quantitative question of the form, what is the time at which x dot is equal to sine x has the greatest speed then geometric reasoning will not be in a position to answer such form of quantitative questions.