Introduction to Nonlinear Dynamics
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Module -04
Lecture-10
1-Dimensional Flows, Bifurcations, Lecture 2

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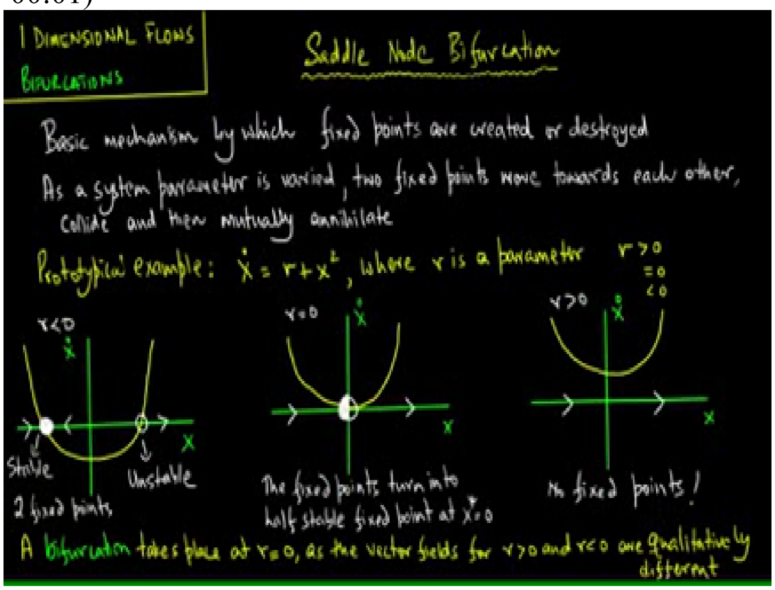

The first bifurcation is a saddle note bifurcation. This is the basic mechanism by which fixed points are either created or destroyed. Essentially as a system parameter is varied, the two fixed points moves towards each other collide and then mutually annihilate. The prototypical example is $\dot{x} = r + x^2$, where r is a parameter which is > 0, = 0 or < 0. So, for r < 0, if you plot $\dot{x}$ versus x, that is the plot for $\dot{x}$ versus x, we find that we have two fixed points.

So, the system has two fixed points one is stable and the other is unstable, for r = 0. When we plot $\dot{x}$ versus x, we find that we only have one fixed point. The fixed point is attracting from the left and repelling from the right. So, the fixed points actually turn into a half stable fixed points at x* = 0, and when r is > 0 plotting $\dot{x}$ versus x reveals that in fact we have no fixed points.

So, a bifurcation effectively takes place at r = 0 as the vector fields for r > 0 and r < 0 are qualitatively different from each other.

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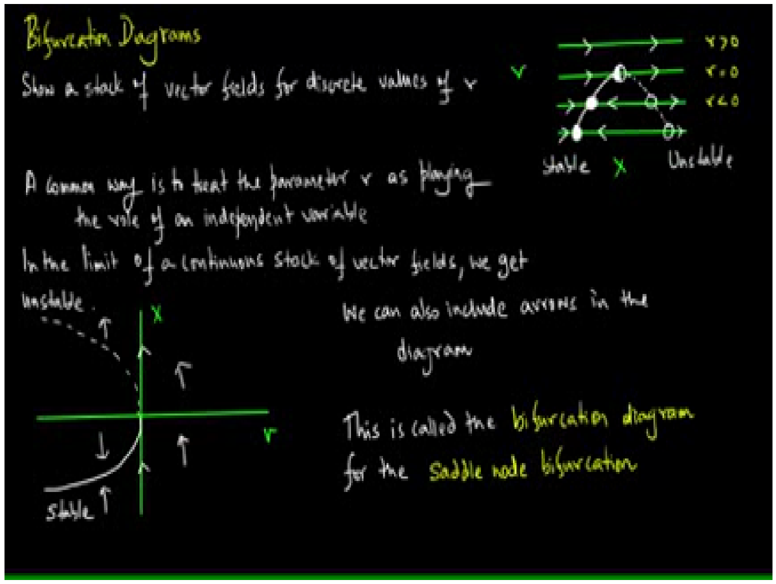

Now, we go ahead and plot something know as bifurcation diagrams. So, if we show a stack of vector fields for discreet values of r that is the stack of vector fields, when r greater than 0 there are no fixed points. When r = 0, there is half stable fixed point and when r is < 0, we have two fixed points, one stable and other unstable and that is what happen when r varies, so we have stable and unstable branches.

So, a common way is to treat the parameter r as playing the role of an independent variable in the limit of a continuous stack of vector fields. What we get is the following, so we plot x versus r as the stable branch and that is the unstable branch. So, we can also include arrows in the diagram, so including arrows in the bifurcation diagram we get. So, this is called the bifurcation diagram for the saddle node bifurcation. So, this is the first bifurcation that we have dealt with and diagram on the left is referred to as its bifurcation diagram.

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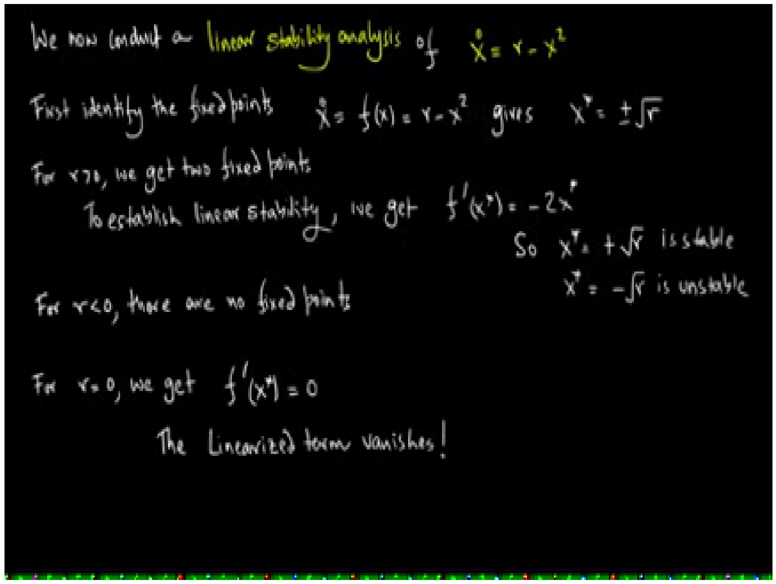

Let us now conduct a linear stability analysis of $\dot{x} = r - x^2$, we first identify the fixed points, so $\dot{x} = f(x) = r - x^2$, yields $x^* = \pm\sqrt{r}$. For r > 0, we get two fixed points and to establish linear stability, we get $f'(x^*) = -2x^*$. So $x^* = +\sqrt{r}$ yields a stable fixed points and $x^* = -\sqrt{r}$ is unstable. For r < 0, there are actually no fixed points, for r = 0, we get $f'(x^*) = 0$ and so the linearized term actually vanishes.

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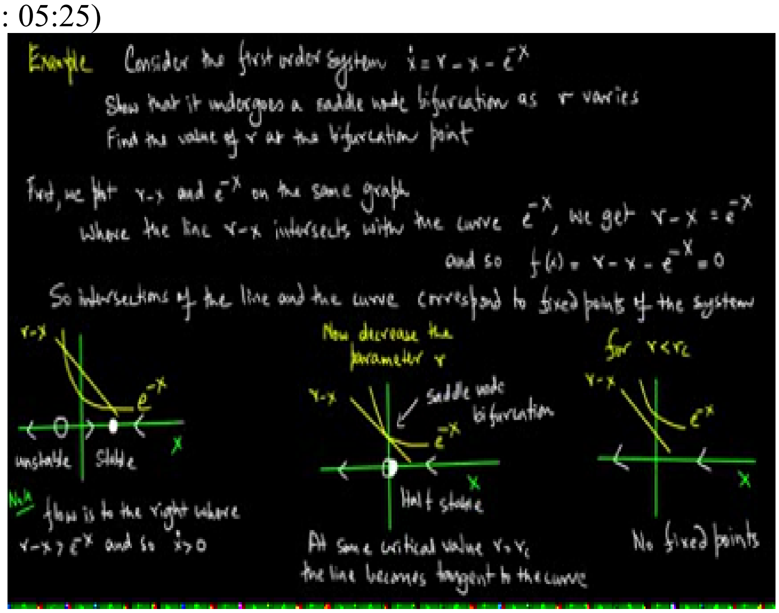

Let us consider an example, consider the first order system $\dot{x} = r - x - e^{-x}$, show that it under goes a saddle node bifurcation as r varies, further find the value of r at the bifurcation point. Now first, we plot r - x and e^{-x} on the same graph, where the line r - x intersects with the curve e^{-x}, we get r - x = e^{-x} and so $f(x) = r - x - e^{-x} = 0$. So, intersections of the line and the curve actually correspond to the fixed points of the system.

Now let us go ahead and make some plots, that is the line r - x that is the curve e^{-x}. So we note that this system has two fixed points, one stable and the other one is unstable. Note that the flow is to the right where r - x is > e^{-x} and so $\dot{x}$ is > 0. Now let us go ahead and decrease the parameter r little bit, so that is the line r - x, that is curve e^{-x} and note that in this case we have one fixed point.

The fixed point turns out to be actually a half stable fixed point. So, at some critical value r = r_c the line actually becomes tangent to the curve. So that is the saddle node bifurcation point, for r < r_c that is r - x that e^{-x} and note that they do not actually intersect at all, so there are no fixed points.

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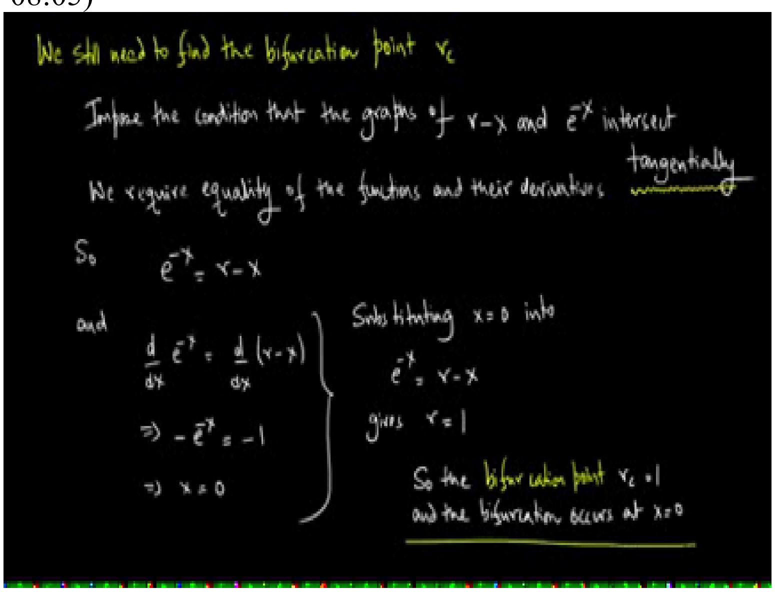

We still need to find the bifurcation point r_c. So, we impose the condition that the graphs of r - x and e^{-x} intersect tangentially. We require equality of the functions and their derivatives. So, e^{-x} = r -x and $\frac{d}{dx}e^{-x} = \frac{d}{dx}(r-x)$ which gives us e^{-x} =-1, which yields x = 0, so substituting x = 0 into e^{-x} is equal to r - x gives r = 1. So, the bifurcation point is at r critical = 1 and the bifurcation occurs at x = 0.

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In this lecture, we introduced the saddle node bifurcation. This is the basic mechanism through which fixed points can be created or destroyed. We introduced prototypical examples for the saddle node. Now essentially what it showed us was the following, we had a model, where we actually had two fixed points, one of the fixed point was stable and one of the fixed points was unstable. And then a parameter in the system changes as the parameter changes, both of these fixed points actually came close to each other.

And at particular point of time they actually merged and became one half stable fixed point and as the parameter changed even more, in fact the fixed point in the system vanished. So, we had a scenario, where we had two fixed points a stable and an unstable. Then we had one fixed point which was half stable fixed point and then as the parameter varied all fixed points in the system actually vanished.

So that is why it's fascinating because when you modelling the real world, you have the real world of which you abstract out a simplified model and that model will have some parameters in it. And so the lesson here is that as parameters vary you would have fairly serious qualitative changes in the underlying dynamics and we illustrated that through the saddle node bifurcation in this lecture.