Dynamics of Physical Systems
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Lecture - 7
Using the Lagrangian Equation to Obtain Differential Equations
(Part – IV)

In the last class, we are talking about the Mutual Inductances and we have already learned how to model general electrical networks. Modeling the elements of the network and modeling the situations like the mutual inductance are now understood. In that situation, we can develop sort of a general technique for handling any electrical network. So, if there is an electrical network consisting of many branches loops and no suppose it is a very complicated network. Situation like that often happen in say power systems. Then the general technique would be first to identify the meshes, identifying the open windows. And the charges flowing in those open windows will designate as the independent variables.

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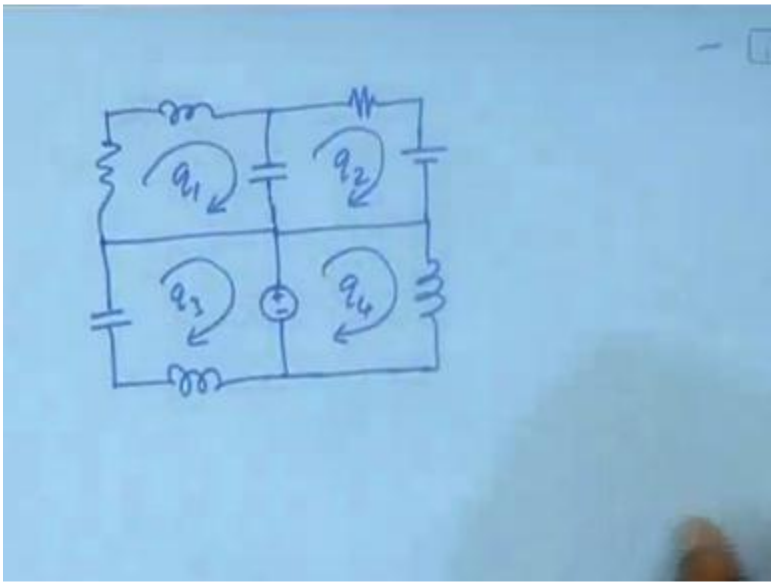

So, if there is a circuit something like this or whatever. That all will do is to say that, here is a window, here is a window, here is and these are the independent meshes and we say q 1, q 2, q 3 and q 4, thus we will identify. So, the identification of the independent variables becomes rather trivial. Now, once we have done. Then, we have to write down the kinetic energy term, the potential energy term and the Ray Leigh function term.

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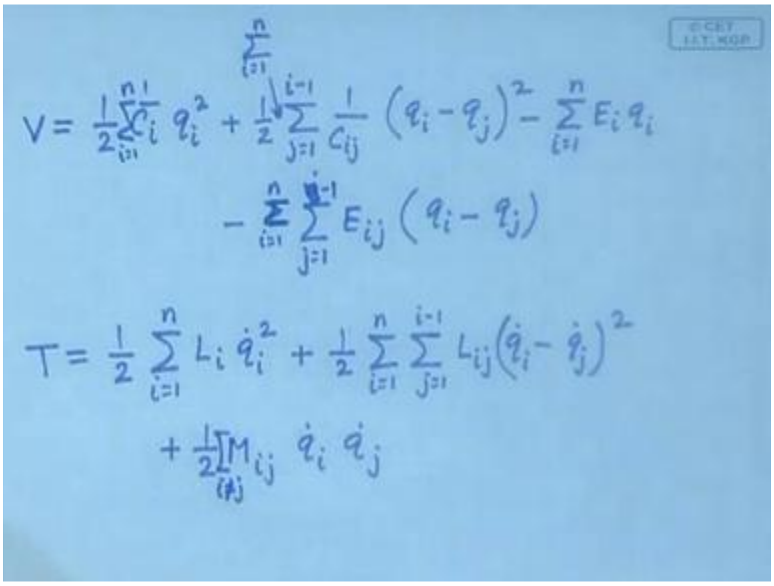

So, these three will be taking the general forms like let us first write the potential energy field will be first the energy stored in the capacitors. So, it will be half C i q i square and that will have to be integrated over all i's. So, i's are the independent variables. So, these are the very i's. So, it will have to be integrated over i is equal to 1 to n. Then, notice that these are situations like this, where a capacitor is placed in basically the charge flowing through the capacitor is this invariable.

But, there can also be situations like this, where the capacitor shared between two branches. So, that will have to be written as again half, we have to sum up over something, I will write that later and it will be sorry, this would be one by know, 1 by c i j. So, this is mutual branch this will be q i minus q j square and this will be integrate from j is equal to 1 to how many it will one less, because I am not considering this self 1, this it is i minus 1.

So, that exhausts the energy stored in the capacitors, plus in the potential term, we also have to include the externally applied forces, which are the voltage sources, so that will be included. So, the voltage sources could be in such a branch also. So, again we will have to individually consider them, it will be minus, again sum over this is E i q i, this is i is equal to 1 to n. There will be one more term, which will where the voltage source is placed in between two branches.

So, that will have to be written as minus, again sum over something, I will write that later. It will be E i j q i minus q j and it will have to be wait, wait did I write it correctly, it has also to be summed over i. So, I will just insert it has to have a summation over what i; that means, it will be i is equal to 1 to n. Similar thing will have to be done here. So, I will just write it once again, the first one would be i is equal to 1 to n, j is equal to 1 n minus 1 or i minus 1, sorry that completes the story about the potential.

Now, comes the kinetic energy T, again has to be written in the same fashion. So, you will have half it will be summed over L i q i dot square, so this is the self branches. So, it will be i is equal to 1 to n plus now the mutual branches, mutual branches means not the mutual inductors, I am saying if a inductor is placed somewhere say here. Then, it will be half, again it will need two summations, over what I will write later L i j q i dot minus q j dot whole square. And this will have to be again summed over in the similar way, i is equal to 1 to n and j is equal to 1 to i minus 1.

So, that exhausts the self inductances, there also be mutual inductances and mutually inductances will have to written as M q i q j, so that will have to be summed over. So, it will be plus half M, M is always i j between which branch and which branch, q i dot q j dot. This will have to be summed over sorry, again in the same way, not in the same way, this will have to be summed over all i’s and j’s, where i is equal to not equal to j. So, that exhausts all the possibilities of the potential energy and the kinetic energy.

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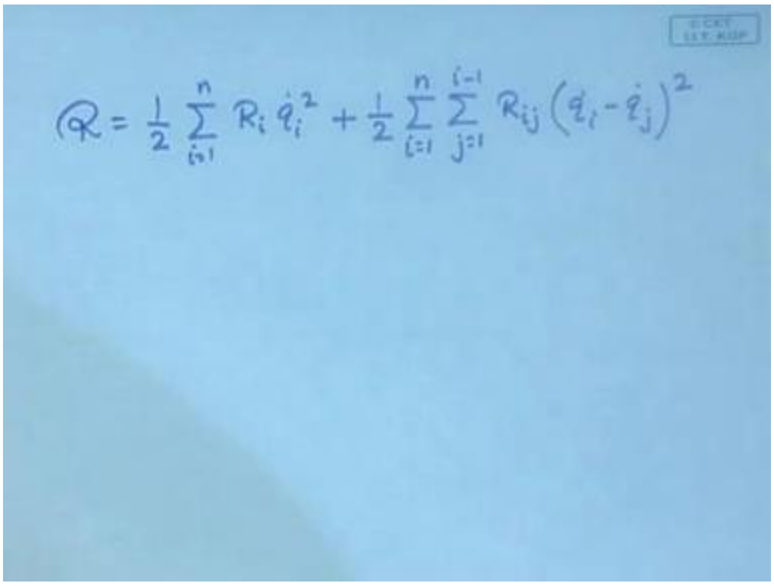

Now, we will have to write the Ray Leigh dissipation function, R that will have the resistances in the self branches and the resistances in the mutual branches. So, we will have to write it as half it will be summed once i is equal to 1 to n, R i q i dot square. These are the self branch resistances, plus half again it will have to be summed over twice, i is equal to 1 to n and j is equal to 1 to i minus 1, then it will be R i j.

The current flowing through this branch would be q i dot minus q j dot square that exhaust, the complete thing. So, if you have a very complicated network, you are will still able to write the Lagrangian equation, exactly in the same way, clear. Now, it is true that often, we do not in our mind we go this way. But, actually when we see the circuit something like this, we identify and then and we write down.

For example, if I have write down the kinetic energy of this one, we will say this L 1, half L 1 q 1 dot square plus half a q L 2 q 4 dot square and so and so forth. Just by an inspection, we can write, you might have notice that in this formulation I left out the current sources that calls for a base based a question, why? The reason is that the current sources need to be treated a little differently.

If we are defining the sources, defining the independent variables as the currents flowing through the meshes, and suppose there is a current source appearing somewhere in one of the mesh, what happens? That particular q i dot immediately becomes equal to that current source, it no long as remain as independent variable.

So, the current sources actually offer advantage and when that happens, when you encounter a current source, you would know that the number of equation that will really need would be one less. That is why, we cannot write down the prescription in the same way, but nevertheless let us look at it how it will happens.

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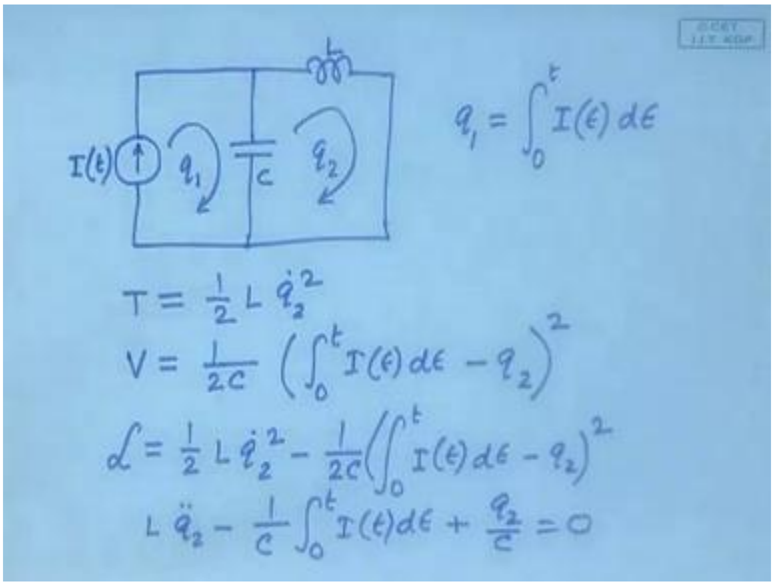

Suppose, there is a current source like this, suppose there is a capacitor and suppose there is an inductor and say only this much, then according to our prescription, what we will say, we will say that this will be our q 1 and this will be our q 2. Now, notice that q 1 dot is nothing but this i and therefore q 1 is integral of the independent current source. So, this is say I over t, it is the integral of the externally imposed current starting from 0 to now.

So, it will be integral of say 0 to t, you have we will have to define some other variable, because here I am writing t, it will be I of say Epsilon d E. Starting from 0 to now, whatever current has flown, that is the total q 1. Now, once we have defined this, the rest becomes exactly the same, why. But, because in that case we will write the kinetic energy here as half say L C half L q 2 dot square, no problem about it.

We will write the potential energy as stored here, it is, why, wait a minute. So, it will be 1 by twice c, but now the charge here is q 1 minus q 2 and q 1 is this fellow. So, we will write this whole thing 0 to t I Epsilon d Epsilon minus q 2, this whole thing square. And your; obviously, the Ray Leigh term is absent here, because you are dealing with the conservative system. And then the Lagrangian function will be half L q 2 dot square minus 1 by twice c, this fellow integral, sorry within bracket 0 to t I Epsilon d Epsilon minus q 2 square.

Now, if you differentiate notice that here, q 1 does not appear at all and therefore, there will be no equation in terms of q 1 that is natural, because q 1 has become dependent on an external variable; it is no longer a state variable then... So, can you write down the equation in terms of q 2, then you will have to differentiate this, this is a independent variable, independent of q 2. So, when you differentiate this fellow, if it appears alone, it goes off.

So, how will you say, the equation will be first d d t of the derivative of Lagrangian with respect to q 2 dot, it will be L. Can you see, yes L q 2 double dot minus now this fellow just expand it, first it will be this square plus this square minus this twice this. So, it will have, whenever it comes to be just this square, it will cancel off, when you differentiate and ultimately what you get, you get one term with this thing. So, it will be twice q 2 and this differentiates with respect to q 2, only this remains.

So, it will be 1 by c integral 0 to t I Epsilon d Epsilon, there will be another term, which will be on these only. It will be plus, because this was plus, it will be q 2 by c, that is how it is. Now, notice that the integral appears in the differential equation that is natural, because you have externally imposed the derivative of q 1. Therefore, if you want to obtain q 1 as a state variable, you have to have the integral.

So, this appearance a bit uncommon, that is why I did not include in the general formalism, but when you have, then you can always do it like this. Now, when we were talking about the general formalism of electrical networks, we sort of took a rather short cut path. You will later realize, why I say it is a short cut path, we said that just identify the meshes and assign state variables to them.

But, as here, I have not bothered to prove that, that will be the minimum set of state variables and in fact in some situations, they will not be where they will not be another thing, we will deal with later. And that is exactly, why later, I will also deal with another different method of obtaining differential equations for specifically for electrical networks. Because, there some additional caution is necessary, but for main ran of circuits, it will be done this way.

Now, let us come to situations, where you need to model some kind of a non-linear element in a system. For example, let us not deal with the electrical circuits, now because one non-linear element, we have already come across that is dry fiction.

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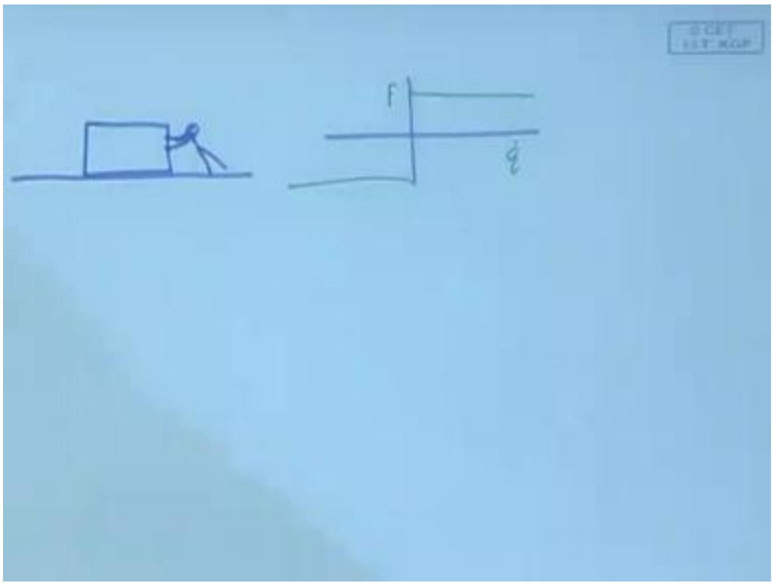

So, where you have got a body resting on a dry surface and if somebody pushes it and then it will initially, it will not move, and then when that threshold force is exceeded, it starts to move and then after that the force due to friction remains more or less constant. So, that is what the character is and we have already seen that, it can be represented by means of a graph something like this, q dot here and a force here.

Force due to the friction, I am talking about, so that is the character of the, this is obviously, it is a non-linear function. A linear function is where it will be form straight line dependence between the force and q dot, here it is not and therefore, it is a non-linear function. How would we represent that, now if a system has this kind of elements or say suppose you have got a circuit, where one of the elements is say a saturable inductor or the capacitance is not really a linear capacitance.

A system where there is a spring, but the spring is not really a linear spring; it is a hard spring or soft spring. In any case, it is slightly bend, the characteristic slightly bend not really straight line. What you will do, then the general technique is first assumed that it is a linear thing. There by what will happen, we will write some kind of for example this one, we will write some kind of a R.

If it had been a viscous friction, we will write R, we will write the equation that way and then say, no, no my R is not then a fixed quantity, it is a variable quantity, we just dependence on these, these, theses, these. Then, we can write it, we can substitute that dependence into the equation, the moment you do that the non-linearity is represented, clear.

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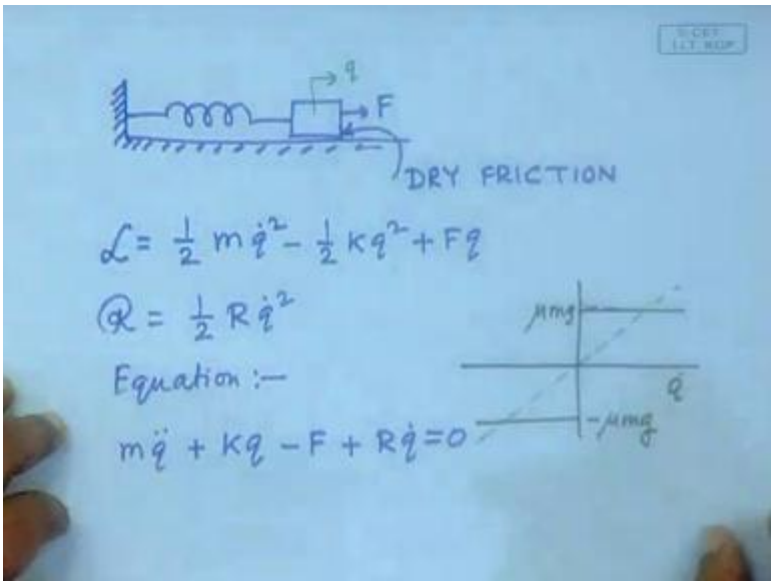

So, let us do a problem with a dry friction element. Suppose, you have surface, there is a spring connected to a mass and where it is pulled with a force F and this fellow is a dry friction. This surface is a dry friction surface. Again, in this case the convenient definition of the state variable would be the q, where q is measured from the unstretched position of the spring that will be convenient.

So, once we do that, we will right the equation as the Lagrangian, we can directly right, because we have already written that once. It will be half m q dot square, for this 1 minus half k q square, for this 1, plus F q. And the Ray Leigh function will be half R q dot square and notice that, here we are assuming initially that this fellow is some number and accordingly we are proceeding to derive the equations.

There is only one state variable q, therefore we can write just one equation, the first one would be in terms of this and it will be, so I am writing the equation directly. It will be m q double dot, due to this minus derivative of the Lagrangian with respect to Q. So, this will be plus, it will be plus k q, for this, it will be minus F plus R q dot equal to 0. So far, so good, it is a simple stuff.

Now, assume that the characteristic of this thing is represented as something like this. So, here is q dot and here is what, suppose you are moving it, it will feel a reaction of m g and there will be a coefficient say mu, so mu m g will be the force created. So, this fellow will be mu m g and this is the minus mu m g convinced. So, in reality it will be slightly up here and then it drops. But, let us assume that it more less that this is not bad assumption, so mu m g and minus mu m g here.

Now, this needs to be somehow represented here, somehow this fellow has to represent that. So, how to do that, tell me, how would you do that, R has to represent instead of, when I write R, I am assuming a function like this, linear function. Instead of that, I have to somehow tell this equation, that my dependence is like this. So, how will you do that, this can be done simply by, because what is happening, mu m g is the magnitude and it is plus or minus. So, under certain condition, it will have to take a plus value, some under certain condition, it will take a take a minus value. That can easily be represented by means of a signum function.

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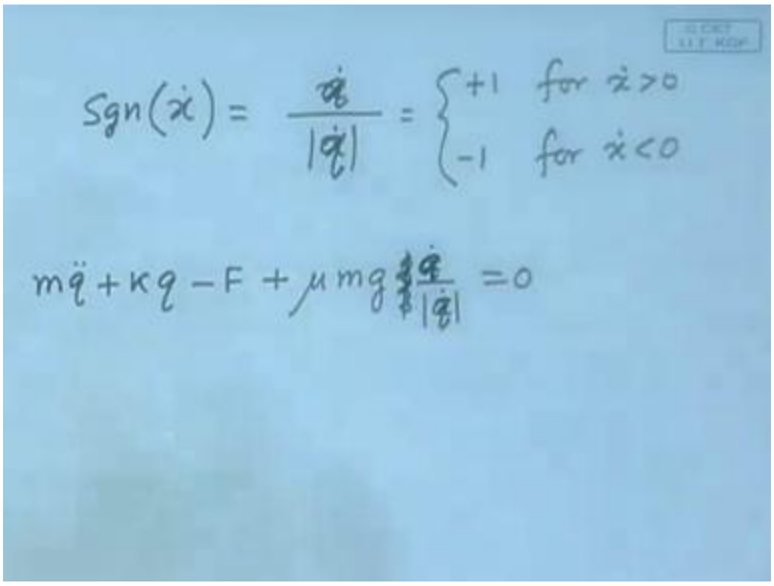

So, we can write s g n signum function of x dot, I have to write in terms of x dot is x dot divided by the magnitude of x dot. So, the magnitude cancels of it is only the sign that remains. So, that will be obviously, plus 1 for x dot greater than 0 and minus 1, for x dot less than 0. So, in terms of that, then we can write the equation that we have derived as it was m q double dot plus k q minus F, all that remains the same.

But, now here R times q dot that has to be substituted to say plus mu m g, the magnitude times x dot by x dot magnitude that is it, q dot and q dot yes, so this would be that gives the equation. So, in general that tells us, that in general if you encounter a non-linear element, whatever method of obtaining the differential equation, will still retain exactly the same method.

We will initially assume that this term is R L C, whatever it is and then substitute the dependence of the R L C whatever it is on the corresponding state variable that is how we can obtain the equations for the non-linear system also fine. Now, the last day I was actually planning to do it as a tutorial, but since we missed it. Let me give you some problem, which you do at home, but they it is to be submitted as an assignment, tutorial normally I prefer to do here, so that I can supervise. But, since we missed one tutorial day, we will I will give you the problem do it and submit, in a separate copy, remember that, fine.

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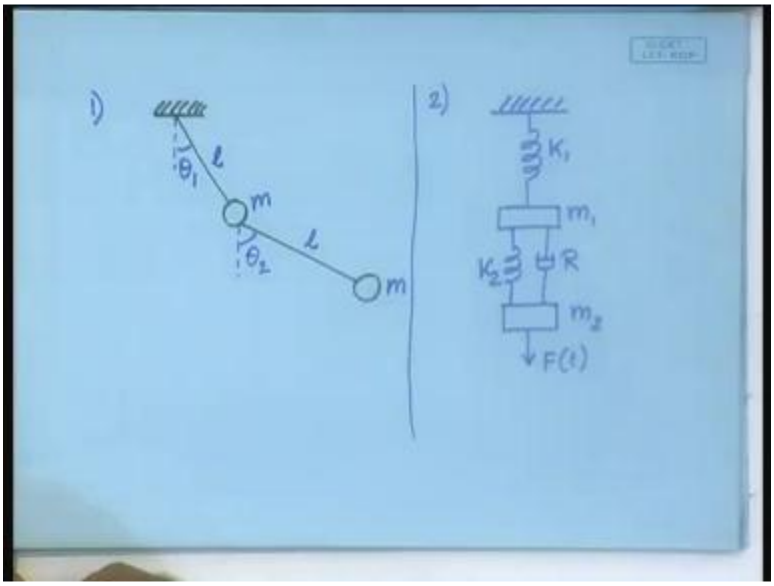

So, let us take this problem, how will you do this, it is a double pendulum, assume for the sake of simplicity that these two l's are the same. Because, I am more interested in you are being able to derive it, than the unnecessary complications. So, let these two masses to be the same that will make it simpler. So, how will you define the state variables, they coordinates the two thetas, two angles, that uniquely define the positional status of the system.

So, this will be theta 1 and then you will have to write the potential energy and the kinetic energy, it is a non dissipative system. So, you do not have to write the Ray Leigh term. Suppose there is system like this, so this problem number 1, problem number 2. A mass is hanging from the roof by means of a spring and then here you have a spring and a damper connected to another mass and this mass is been pulled downwards with a force F of t.

So, how will you adapt this problem, how will you conveniently define the variables q 1, q 2, but that q 1, q 2 how do you define? There are two ways; obviously, you allow it to hang vertically. And then wherever these two masses are call it q 1, q 2 that is one possibility. The other possibility is no, no do not do that, you raise this fellow with your hand, so that ultimately, it goes to a position, where this unstretched.

Hold it there and now raise this one, so that this one also goes to unstretched position and call that, these two positions of the masses as q 1 and q 2 or as the 0 positions of q 1 and q 2 and measure from there. Both these possibilities are there and you define according to what you find convenient, both are the same actually, both are equivalent actually. In one, you will have one additional term, which will cancel off, when you differentiate.

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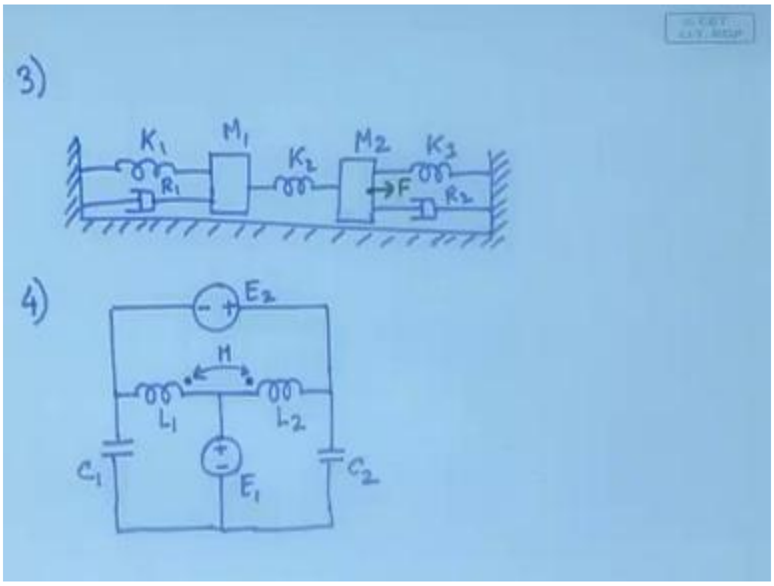

How about this problem 3, there are two masses, this one is connected by a spring damper arrangement. This one is also connected by a spring damper arrangement and these two are connected by means of a spring and say there is a force F, pulling this mass. In that case, how will you define the state variables, again there are two masses; their positions uniquely define the systems, positional status.

Then, how will you conveniently define that, may not be possible, in this case that may not be possible, in the earlier case it was possible, but here since these are two rigid walls, this may not be possible to bring this springs to unstretched positions. So, in that case what, you will bring two of them to unstretched position and allow one to stretch. When you start it, just define a q; that means, is already stretched to this extent and then start from there.

And once you define the states variables, it will not be very difficult thing to write down the kinetic energy and potential energy, this step itself is not very complicated. Let us gives some circuits, call them k 1. Now, take a circuit, so here you have L 1, L 2, C 1, C 2. Now, you can say E 1 and E 2, what will you define as state variables. Let us add some more complication; there is mutual inductance between these two, oriented like that with the dots.

So, how will you define, simple q 1, q 2, q 3 and then follow the same logic that I showed, this exactly can be done by that, fine. Do you any other circuit problem or this more or less gives you practices. You asked me a question, that day, that in case of the pendulums tip is being or the suspension is being moved, then does not they define the another state variable. For that system, where they want that was considering at that time, it was independently been moved. And therefore, that does not become a state variable, but suppose that is not a dependent one and take a situation like this.

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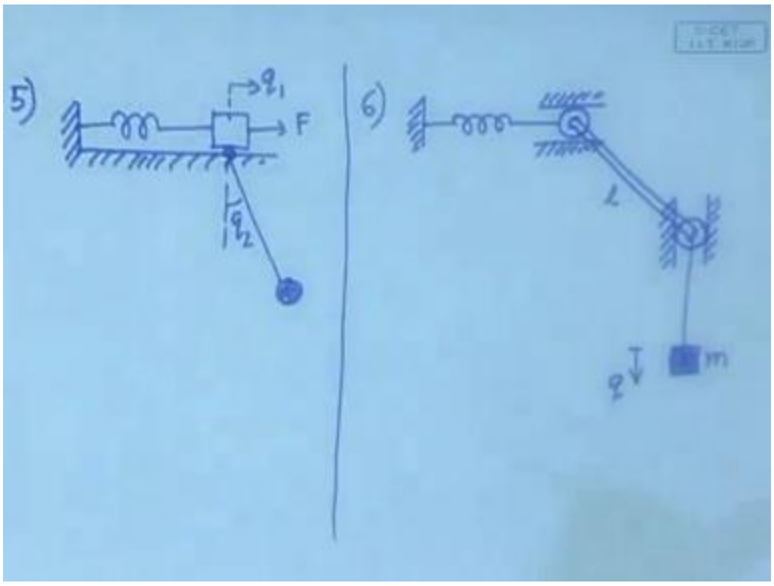

So, here you have a F, and then from here then what, so you have got a mass in a simple harmonic motion. But, it will not remain a simple harmonic motion, because there will be another thing, pulling it or pushing it or whatever it is. So, it will be another application of force, but can you see that, it becomes then quite complicated stuff. But, if you try to write down the differential equations, it will not remain quite complicated, it will be simple, but how do you do that; let me see.

In this case, what will be the state variables; yes q may be position of this one. Now, you can define it as the deviation from the unstretched position, so this becomes a state variables say q 1. And or maybe you can call it q 2, which is theta, does it uniquely define the positional status of the system, is it does. Notice that, one is a translational variable; another is a rotational variable, generalized coordinate that is the concept of generalized coordinate. Whatever, you want to define as the minimum number of coordinates that defines the positional status of the system, fine.

Now, can you write down now the kinetic energy and the potential energy of this fellow with respect to this q 1 and q 2 try this, it will be good? So this was problem 5. Let us think of a mechanism, where you have a rigid wall to a spring, but at this point, there is a wheel, which is able to move in between this and there is another guide. So, there is a guide here, which forces this thing to move this way only and there is another guide which forces this thing to move this way only.

And suppose, there is another wheel and this two fellows are connected by and then you have got, say a mass hanging here. You see, the mass is hanging here and you can apply a some kind of an initial force to this one to see what will happen to it. As you pull it, this will come down, this will come this way, the spring will stretch and then it will go through a some kind of a motion.

How do you write the differential equations for that, what are the state variables, q 1 is what, assume that these two are mass less and then what displacement of the mass. Say and then

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Do, you really need that…

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Yes, only q 1 will satisfy the requirement, because this position is uniquely defined by q 1. So, the task is to define the minimum number of variables necessary in order to define the positional status and that can be done only with one variable in this kind of system. So, once you have written down that, then you have to q 1 may be the displacement something like going like this q. And the on the basis of that, you have to specify this as it.

Try in a Newtonian way, it will become enormously complicated, try in a Lagrangian way, it will become enormously simple. Similarly, you had considered a pendulum, where this fellow is moving, this way. Can you imagine a system, where this fellow is moving that way; yes you can always do that say like this.

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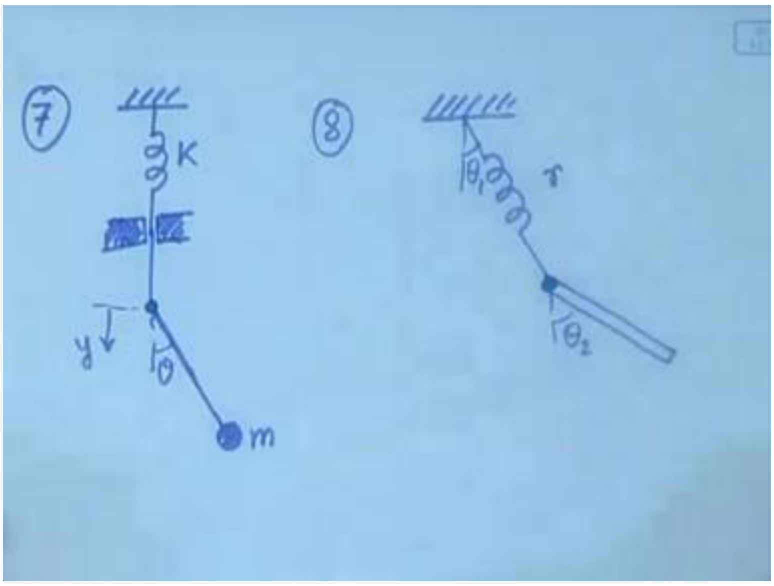

You have got a spring and there is a say guide and this well as coming, so that this is not allowed to move this way, this only allowed moving that way and here you have a point of suspension from which the pendulum is hung. So, this is a guide that allows it to move only up and down. In that case, how many variables would you need, think 2, which 2?

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So, the vertical displacement of not the mass of this point, because mass is here. This is just a point, where is being held. So, point of suspension and this point of suspension are going up and down. So, you can say this position is one state variable; the other is obviously, theta. So, you can say, if it is a x coordinate, it is a y coordinate, then this can be simply y from here and it is possible, then to define y 0 as the unstretched position of the spring, this would be problem number 7.

You have done a problem with the spring pendulum; let us complicate your life a little bit. Here, you have the spring pendulum, but at this point, what is hanging is a bar, so yes you will have to consider, it is centre of mass and then you need only two variables. Though, it is a far complicated system, two variables, no, because if this were not there, then because it is a spring, you would have to consider the radial motion, as well as the angular motion.

So, for this point you need two and again for this one, there is another additional angle, three variables, you need three variables. So, you can say theta 1, theta 2 and again the same way you have to define. As I told you, that there are many possible ways of defining this, how it define is your business in the problem, that I did probably I had taken the vertically hanging downward position as the data. So, that is the one possible way. So, there has to be a say r direction in this side, but how you define, it is your business.

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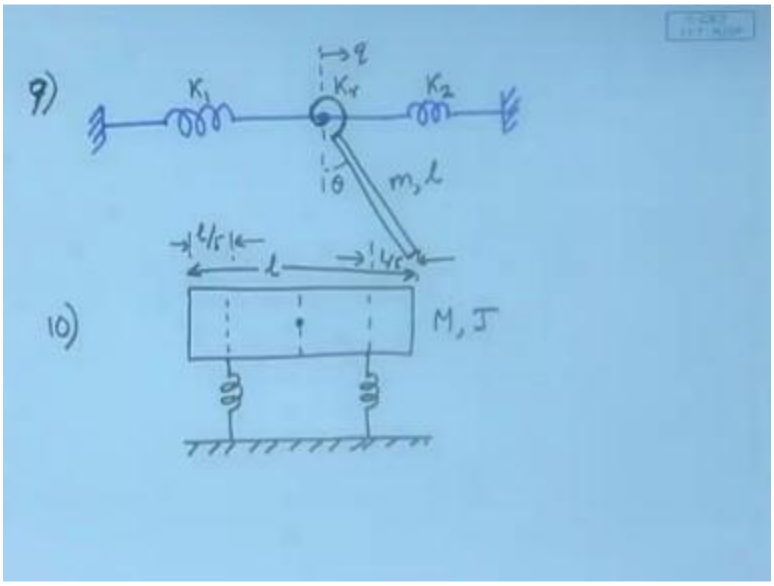

How, about this sum of the clocks, have it like this a spring, another spring, so there are two springs and it is able to move like this and there is a tortional spring that connects a rod. So, this is a tortional spring that also gives a springy effect. So, how will you define in this case, you can say this is k 1, this is this one is k 2 and this is a tortional spring say k r rotational spring and then it has a mass m length l.

Obviously, you can define 1 coordinate theta, what other you need, so the position of this. So, you can say, can you write down the kinetic energy and potential energy in terms of this would be a little tough, but nevertheless, not it not a very formidable challenge. What problem 9? I will give you 10 problems; the tenth problem is where you have, say do you know, how cars are balanced on the suspension.

Suppose, you have got a car, which I am representing by means of a very badly drawn rectangle, no one would like to get into such a car, but nevertheless imagine that is a car.

And I will not draw wheels, because all I am tried to depict is that, while the wheels move, they are on a bumpy ground. So, the bumpy ground is giving some kind of a force this way and this fellow is moving, so that is what you feel all the time, when we move in a car.

And these act at specific positions, where this can be represented this suspension can be represented as springs. At this stage, I will not make it complicated by saying that this two position are being pushed up and down. What, I say is, just let it rest and then give a push somewhere here, it will oscillate and move. Now, it has the point is that you have to find that how many degrees of freedom it has.

Obviously, I will need to mention, where is the center of gravity, I will need to mention where these two suspensions are, fine, let that be that be given. So, the total length is l, say out of that these two are symmetrically placed, this is l by 5 and this is also l by 5 and the whole thing has a mass m. You will also need the rotational motion; there will be a rotational motion another degree of freedom. So, you will also need the moment of inertia there.

So, what are the degrees of freedom and how is it allow to move, just one mass remember, but the same mass will now have two degrees of freedom, it can go up and down, it can also rock like this. So, in order to define the positional status of the system uniquely, how many variables do you need, 2. One represents this position and represents that and the elongation of the springs will have to be measured from both, it is not just this.

So, if there is a tilt here, so this fellow this particular spring is compressed and this particular spring is elongated, so that will have to be taken into account in order to measure the potential energy in the springs. So, the problem is that just a bar resting on springs and if you press one, it will go on doing like this and that is what we are trying to find out later. But, first job is to obtain the differential equations, clear, do you able, fine wonderful.

So, I have given you 10 problems and that 10 problems solution have to be submitted in the next day, no not tomorrow, next week, I will give you the week and in between. Now, you have noticed, that whenever we were trying to use the Lagrangian method to obtain the differential equations, it always yielded second order differential equations. Obviously, m q double dot that was one term.

So, whenever you differentiate with respect to time the derivative of Lagrangian with respect to q i dot, it obviously, yields a double derivative. But, then probably you have also done some courses on numerical methods, where you have solved the differential equations. Have you solved second ordered differential equations, how did you solve; you first brought it down to first order.

So, obviously, then after you have obtain the differential equations this way, you need first to bring it down to the first order, if it is a one second order differential equations, you define two first order differential equation and so on and so forth. And, then only you apply those techniques. So, even if you are trying to solve it analytically, then also it is always convenient to have it in the first order form.

Though, I will not say that techniques do not exists to directly solve the second order equations, there are, but nevertheless. Since, we are very comfortable, we know the techniques in the first order; the next step will be to bring the equations down to the first order. Normally, how do you do that, you define an additional variable and that additional variable will can be anything, can be what, what will you normally define as the additional variable

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Q dot may be, so you have m q double dot, you say q dot something, you do not do that. But, then that will not prove to be very convenient algebraically, why because it is always m q double dot. So, if you say m q dot is a variable, then it will be algebraically far simpler to write it down. So, in general that is the technique of deriving the first order equations, we define what are known as momentum. Momentum as the other variable not the q dot; that is the standard procedure, which will make things very simple, I will show you.

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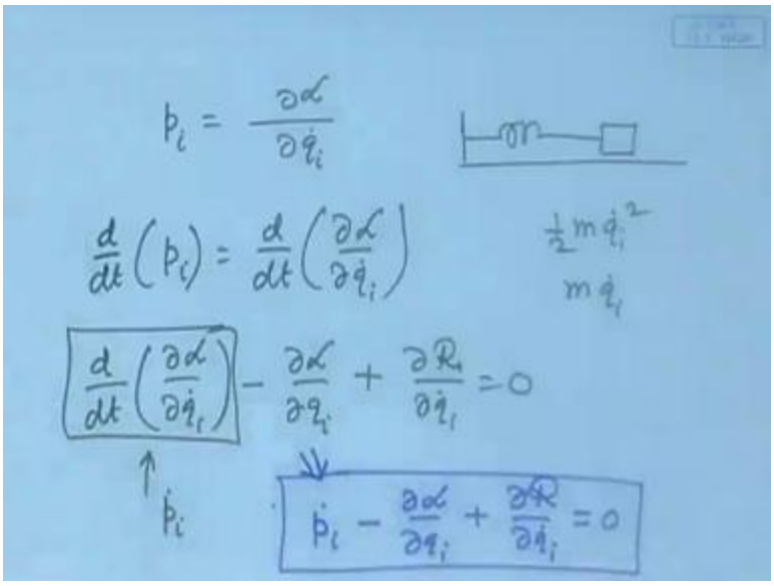

So, first we define the p I, notice we had q i and now we are trying to define p I, what is p I, notice what is it, the momentum in the generalized i direction, that is nothing but that is it. Imagine any of the situations that we have come across, say there is a mass, what was your Lagrangian, Lagrangian had this term m. You differentiate with respect to q i dot, it get that is, what p i is.

So, we in general define, the p i as this, for the trivial cases, where this are you can easily identify this no problem. But, you will see that the advantage, when there are coordinates coupled, I will show you. For example, the mass connected with a spring as a pendulum, we have seen that there the kinetic energy content, both these things radial as well as the, in that case, what is the momentum simply.

We already had the Lagrangian differentiate with respect to r, differentiate with respect to theta, differentiate with respect to r dot and theta dot, so that gives. So, once you have this, you can write then d d t of p i is nothing but the first term in the equation that we had, so our equation was R not i. Now, notice this term becomes, so you can directly write the first order differential equations. So, finally your equation becomes p i dot minus this, so it has this dotted term, so you directly obtain the first order differential equation. Now, I will illustrate this with examples in the next class that is all.