Dynamics of Physical Systems
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Lecture - 10
Obtaining Differential Equations Using Kirchhoff's Laws

For the last 2, 3 classes, we have been learning how to obtain the first order differential equation. So, before going to the electrical circuit specifically, let me give some problems that you solve before coming to the next week's classes. Firstly, all the problem that you are done using the Lagrangian method, where you have obtain the second order equations, you obtain the first order equation for them. So, that is the think to be done.

But, in that assignment do not do the whole problem all over again, because you have already drawn the physical system diagram obtain the kinetic energy, the potential energy, the Lagrangian all that remains. So, in this assignment, what will do for those problems, you just start from there, and then do the rest of the steps to obtain the first order equations. In addition to that I will give some problems will you solve apiary from first principles to obtain the first order equations.

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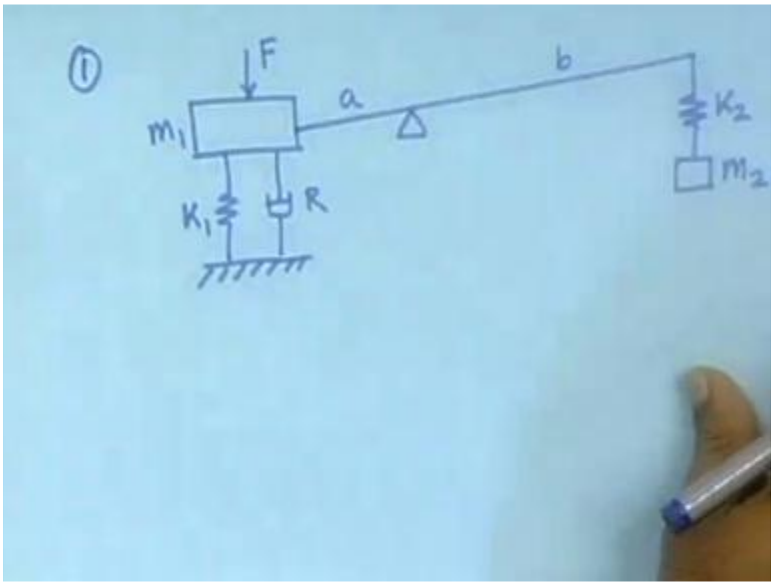

The first problem will consist of a mass, which will be standing on a spring damper support on the ground and if we apply a force that will move up and down depending on the force. But, that is not is the complete story, then is a lever and at the other end of it, there is a mass hanging by a spring. So, there are two masses m 1 and m 2, there is the spring constant k 1 and there is another k 2, there is 1 r and there I have would be also specify the ratio a and b.

For the sake of simplicity, assume that this rod will remain close to horizontal position. So, do not really need to consider this angle that approximation you assume and go ahead. So, in this case there are two masses and therefore, there will be two momentum coordinates and two position coordinates. Four dimensional system really and then the rest will be simply, is it simple, you do it, fine, so that is one.

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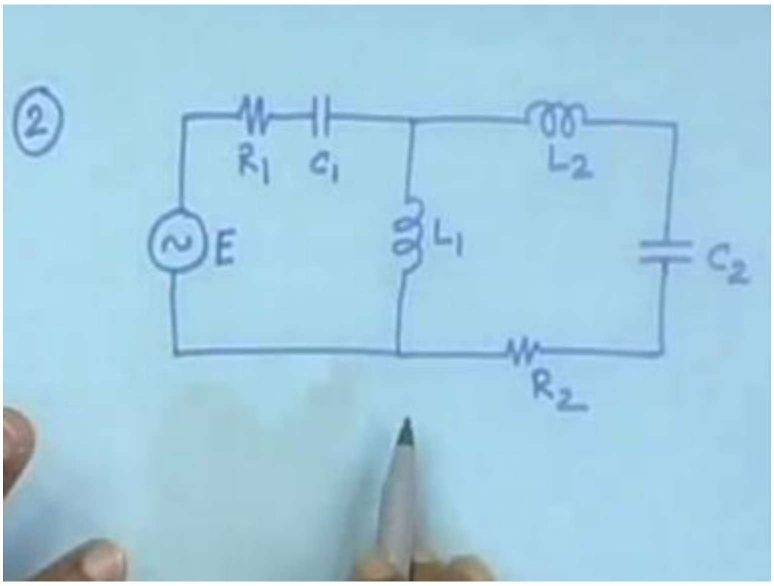

Two, I will give one electrical circuit to be done using this method. There is a voltage source connected with a resistance and a capacitance and then an inductor. Then, again an inductor and then again a capacitance, I would like to add something more, fine. So, L 1, L 2, C 1, C 2, R 1, R 2 and E, for this obtain first order differential equations, note it down.

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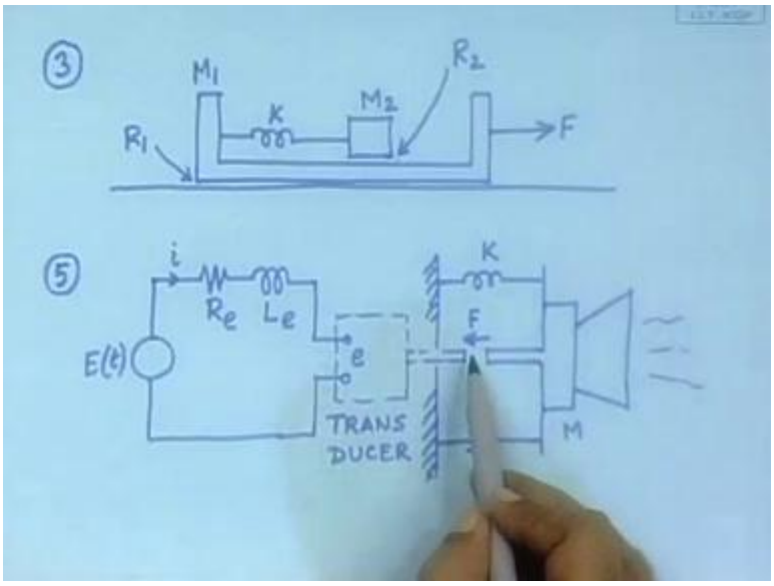

Suppose, there is a cot like think, which is able to move on the ground and this fellow has some mass say M 1 and it is being pulled by a force F. And on this surface, there is a friction, that is say R 1 is consolation, but here is a mass, that is connected by means of a spring, so that is another mass M 2. And here is the K spring constant, I suppose that complete the story, no, there will be some kind of a friction.

So, how many degrees or freedom does it have, two position coordinates and two momentum coordinates. Because, now we are talking in terms of the first order equation, when we do, so we will have to stock in terms of both position as well as the momentum coordinates. We have already done the problem in the class for the inverted pendulum; obtain the first order equation for that.

So, I am not again writing it, we are already done the problem, but we did it following the Lagrangian method, so obtain the first order equation for that, so that is problem number 4. So, I will write problem 5 here, this problem will concern the way, the loud speakers work. So, how does the loud speakers work, it is a electro mechanical system, loud speaker is an electro mechanical system, there is a magnet, which is energized and that pulls a diaphragm which oscillates.

The diaphragm pulling another stuff that is the mechanical system, the other thing is electrical system and in between there is a coupling two magnetism, which is similar to the decimation problem that we did, so remember that. So, you have some kind of a voltage source, so that is E t, the signal which you want to be reproduced in sound. A resistance and inductance will be there, because there is a winding, winding will always have these things.

And then you imagine that you apply this to these two terminals and here there is some kind of a transducer that applies, that pulls it with the force F, here is an iron piece, I am just symmetrical drawing, it should not like that, but nevertheless. You understand that here is something that converts the current signal into a force signal, to be applied on the mechanical side.

What is there in the mechanical side? There must be some kind of a mass, which is essentially representing the mechanical arrangement of the loud speaker. So, get the sound here, fine, but not only that this fellow should not be hanging in the air that has to be attach to some kind of a ground by means of a spring damper arrangement. So, this is the mechanical ground and this arrangement is connected to the mechanical ground by means of the spring and damper arrangement.

And then as this force is applied, it will oscillate, it will vibrate and that is what is communicated into there. So, there will be a mass here M, there is a spring constant K, there is a resistance of the device RD may be and here is the R electrical and L electrical. And here is the transducer is there anything I am missing, no and here is the current that is I flowing, fine.

So, how you will do it, as I have already shown in the decimation problem, this can do in two possible ways. In first, you take the electrical sub system and the mechanical sub system separately, write down the separate t, v Hamiltonian and all that and then obtain the differential equation separately. And then since you know how much is the force depending on the current here and this also applies some kind of a back e m f, say e always thus so… Whenever, there is an electrical to mechanical coupling, it will always happen.

So, it also applies a back e m f, because here is a here is something magnetic going on, there is an interaction between these two, that will apply a voltage here, which will seen as a back e m f. So, here is a voltage that is seen by the electrical subsystem. So, the total amount of voltage applied on the electrical sub system is E minus small e and that one the mechanical system is F.

If you want to do it as a single electro mechanical system, then also you can do that, but remember in that case the F is proportional to I and therefore, it it will result in a velocity dependant potential. So, you cannot take the original Lagrangian equation that we derived, we have to take the enough for the first principles, we have to take t instead of l, that is all, we have I have already illustrated, how to do that.

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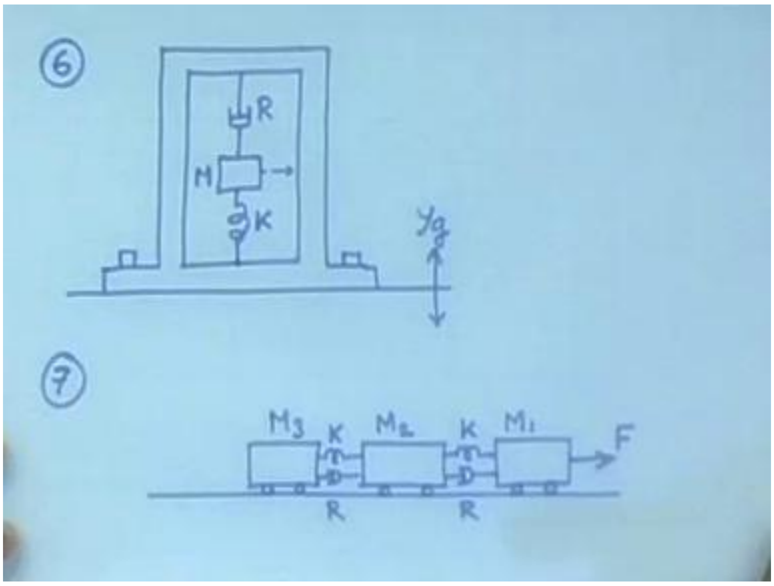

This sixth problem is ever seen a seismograph by which the ground oscillation is measured. So, it is a similar kind of device, say here it is routed to the ground and the whole thing is a hollow box. Inside it, you have a mass spring damper kind of arrangement. So, when this ground moves, this body cannot move instantaneously.

So, there difference essentially measures, there will be a pointer of like this, which measures the up and down motion of the ground. I mean, this is rather crud picturization of what is actually inside, what is actually inside this firm of complicated, but this gives you idea about how to do it.

Now, here you have the ground motion, ground is in that case not static, there is a ground motion, say in the y direction and then this is K and this is the R and here we have M and this fellow. Well, if you considering the motion of this one, you do not need the need to consider the mass of this, because the whole thing is moving up and down. So, you can see that this point is moving with that velocity. So, this point and that point are moving with their velocity in between you have to modulate.

Problem 7, the train, suppose here is the track, here is the engine and here are the two coaches. How are they connected, there always connected by a spring damper arrangement. So, you have 3 masses M 1, M 2 and M 3, there are two springs, let them be equal, so K and K. There two dampers, let them be equal R and R and obviously, this fellow is pulled with a velocity with a force F that is the engine force.

Force providing by the engine, so make a model of that, they will all individual have their own degrees of freedom. In actual practice, there is a limit to which these things can be compressed. In actual practice, there are two hard plates which can collide with each other, so compression is limited, but they tension is not limited and this compression be limited is what, it is a non-holonomic constraint.

So, if you have a non-holonomic constraint, then also the same set of differential equation has to be obtained, so that is what you obtain those differential equations. So, that gives you reasonably good amount of practice in modeling, in order to obtain first order differential equations. Now, we promised to take up specifically the electrical circuits, today where the basic law would be the Kirchhoff's law.

By the way, some of you may have heard the name as Karchoff, it is not Karchhoff, it is Kirchhoff. The guy was German and German all the letters have pronounced as they are. So, you is always ooh, bus is bush. So, Karchhoff is not the right pronunciation, Kirchhoff is the right pronunciation. So, the two Kirchoff's laws, current law and the voltage law, what do they say, what do they say. If you have a loop in that circuit, then you have the voltage law, it which says that as you traverse around the loop, the total voltage is 0. Some of the voltages, it traverses around the loop is 0.

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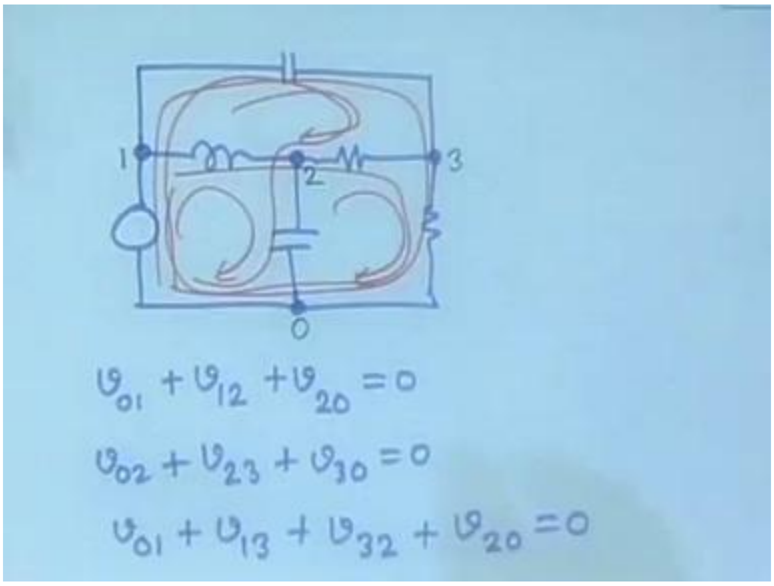

So, suppose you have circuit like this, then what are the loops, well I can identify easily when a loops. This is a loop, this is a loop, this is a loop, this is a loop, this is a loop, all the loops and along around all these, the total voltage should be 0. So, that is the point number 1, that the Kirchhoff's first law. The voltage law says that along, around all the loops the total voltage should be 0.

So, if you can identify the by names, say if I say this is point 1, 0, 2, 3 then we can easily write down the positions of the loops. For example, this loop would be said as 0, 1, 2 loop, this loop would be 0, 2, 3 loop. By the way, there is another concept that we to have identified these are the nodes. This is the node, what is a node, nodes are essentially the points, which can at least in theory be are different voltages.

So, I can see that there are 4 nodes and these 4 nodes, I have marked as 0, 1, 2, 3 and the individual elements are in between them clear, fine. So, if you write it this way, then what are the possible KVL equations? For this loop, it is v 01 plus v 12 plus v 20 equal to 0, notice I am not writing the opposite things, I am writing 01 not 10. The idea is that you have traverse always in one direction, which direction is choose is your business, you could as well as choose the opposite direction, it will not matter. But, you always have to choose a particular direction and go along, around it accordingly.

So, similarly you can write v 02 plus v 23 plus v 30 is equal to 0, you can also write for this big loop, you can write v 01 plus v 13 plus v 32 plus v 20 equal to 0 so and so forth, you can identify many loops in here; at for all that the Kirchhoff's voltage law would be applicable.

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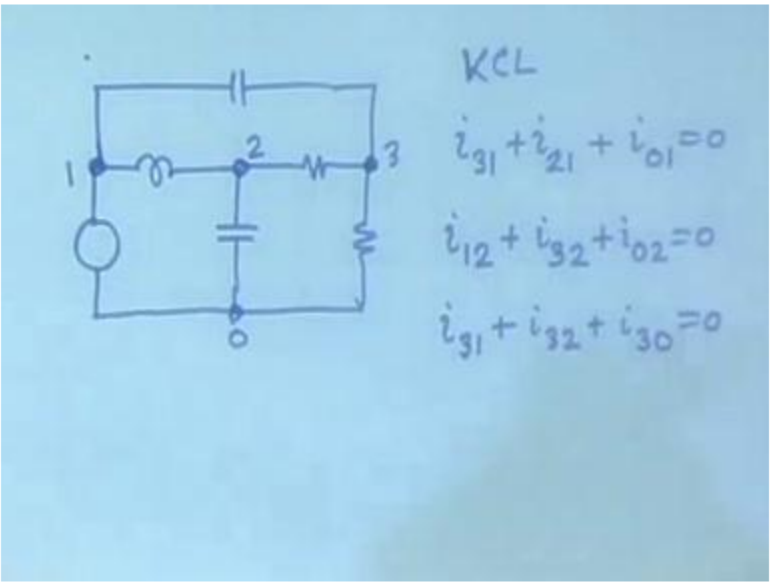

What is the Kirchhoff's current law? They will be applicable to, let be draw the circuit once again, we are started with in that Kirchhoff's current law and you have to identify the nodes 1, 2, 3, 0. And then the currents that either enter or exit, each of these nodes would sum to 0, which means that the currents that are coming into this node, which are this current, this current and this current or the ones at exiting node, this current, this current and this current, they will all sum to 0.

So, you can write the KCL equations as say I am assuming the currents that are entering a node to be 0. We have made, that is my conversion and how I take it is my business. So, I consider the once that are entering the node as 0, so I will write, what, no I have not said that yet, all I have said that is that, there is a direction of current here, there is a direction of current here, there is a direction of current there. All of them add to 0, it does not mean that actually flowing like this, they could flow like that.

But, when I say i 31 plus i 21 plus i 01 equal to 0, essentially have considered a specific sense of positivity of the currents. If the currents actually flowing in the opposite direction, there will be negative, so what? So, I can write similarly the KCL equations that is for node 1, I can also write for node 2, i 12 plus i 32 note not 2, 32 plus i 02 is equal to 0.

Here, now I can as well write the equation as the currents that are exiting the node, interchangeably. So, for 3, I will write it as if there all exiting the node, so it will be i 31 plus i 32 plus i 30 equal to 0, that is all right. So, whether you assume that they entering direction is positive or exiting direction positive is up to you. You easily see that, if they are all negative, they will cancel off.

But, now where had we seen that these are the voltage equations and these are the current equations. Plenty clear that the number of equation that will have is for more than, what we really need, so something is to be done to organize this mesh. Secondly, how to write the differential equations, in order to write the differential equation, there has be clues, something like differential has to be there.

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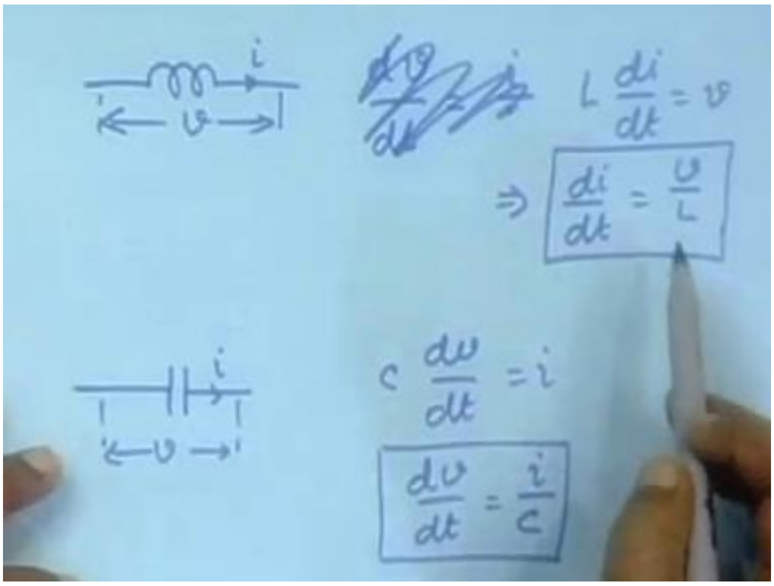

That clue comes from the fact, that if there is an inductor in a specific line and the current flowing through it. Then, we can write the d d v d t as for this L d i d t, so L d i d t is equal to v, so this gives d i d t as v by L, so here is the voltage across the inductor. Similarly, if there is a capacitor in a line, again the current is i and the voltage is say v, then it will be C d v d t is equal to i or d v d t is equal to i by C. So, these are; obviously, differential equations.

So, the immediate conclusion is that, if in the circuit I notice, that there is an inductor and then I will be able to write this equation, only for that inductor. And if there is a capacitor, I will able to write this equation only for the capacitor. These are first order differential equations, so that least to the conclusion, that for each of the inductor there will be one first order equation. For each of the capacitor there will be one first order equation.

What to do about the right hand side, this could or could not be a chosen variable. So, all we need to do express this in terms of the variables using the Kirchoff's laws. Let us illustrate, let us take the as yet we have not organize this mesh too many equations. You might say that why not go the same way as we did, in case of the Lagrangian equation or the Hamiltonian equation use the mesh. That means the open windows, in this case all these loops are not the independent loops, only this and this remains the independent windows, yes you do that, so that will be the Mesh current method.

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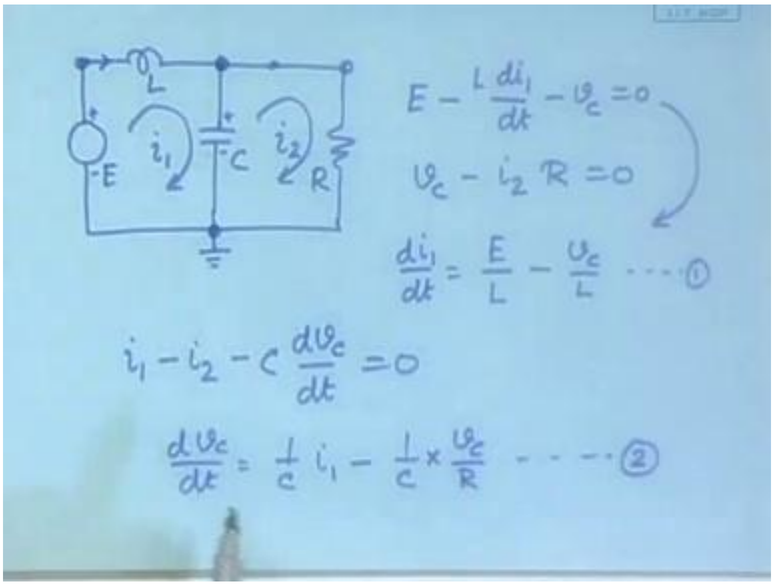

Let us apply that to first a very simple circuit and see the consequence, this one I am doing, because I already did for the Hamiltonian method, so that you can compare. This is E L C R and we said at the time that here is a loop, which will called q 1 and here is a loop which will called q 2. But, while writing as the Kirchhoff's laws equation, I will give i 1 and i 2 currents, so i 1 is q 1 dot i 2 is q 2 dot fines.

Now, what are the nodes, here is a node, here is a node, here is a node, here is a node, wait, they could all logically be a different voltages. But, this is trivial, why because there is only one branch connected to this and going on to this. And therefore, the Kirchhoff's current law will be a trivial equation, so will ignore that. So, effectively there a two nodes and whenever use that we always assume that one of them is at the ground potential, so that we measure the voltage of the other, so that is the standard procedure.

But, let us see; first let us apply the voltage law. Mesh current method we say, in the first loop the voltage law will look like E, E minus here it is, it is a voltage drop L d i d t minus this is the voltage rise, so instantaneous polarity. So, this is voltage rise E minus L d i 1 d t, here it is, if this is the assumed positive polarity. So, you have to assume, which is your conversion positive polarity, which is conversion of negative polarity, so assume that this is my polarity conversion.

If the upper plate is positive with respect to the lower plate, I will call as positive. So, if that is, so then this also minus, fine and the second equation gives v c minus i 2 R equal to 0. The first equation immediately gives the differential equation d i 1 d t is equal to take it to the other side E by L minus, so that is first equation, no problem, done. But, then this is not a differential equation, the second equation is not a differential equation. Therefore, we have made no head way in obtaining complete state of differential equation using the Mesh current method.

Then, I have to apply the Node voltage method; I am assuming that you have learned all that in the first year. So, I am not redoing the whole concept, I am just using the method. So, then we have to apply the mesh node voltage method, because the mesh current method proved to be insufficient to obtain the complete set of differential equations. So, if you apply the node voltage method, how many nodes are there 1, 2, 3, but these two are trivial, I will apply to here.

For this one, if I say the node voltage method, then it will be the Kirchoff's current law, will be i 1 here, i 1 here minus i 2 here and here is minus, what is this current C. Yes C d v c d t is equal to 0, which immediately gives you the differential equation d v c d t is equal to 1 by C i 1, i 1 is state variable. So, it remains minus 1 by C i 2, i 2 is not because I am now obtaining the differential equation in terms of i 1 and v c. So, I need to do something about it, but this one helps.

So, I can see that this equal to i 2 is equal to v c by R, so into that is another differential equation. So, the crux of the story is that, yes it is possible to obtain the differential equation starting from the Kirchhoff's laws. But, you have to apply both the Kirchhoff's law, you cannot obtain simplify the node voltage or simplify the mesh current method, you have to apply both, clear.

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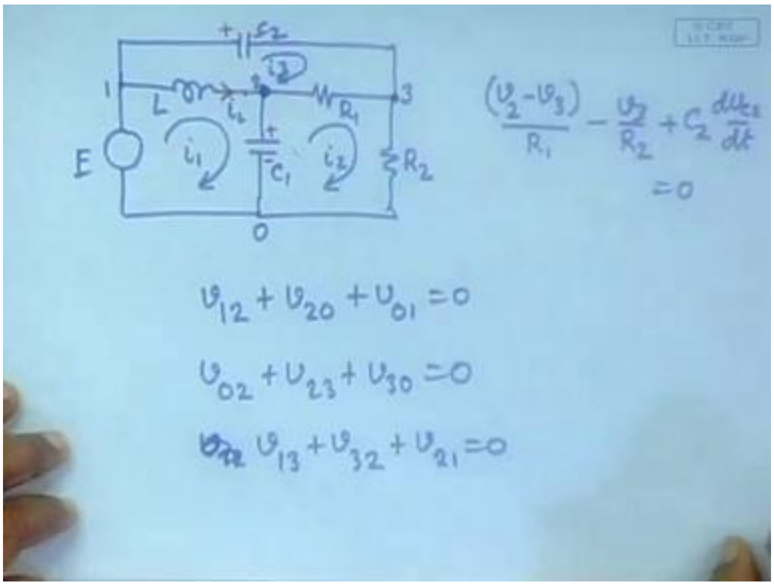

Just let us illustrate it with another example to get it clear, the same example that we have we are doing the 3 loop example, it is a E, then an inductor, then a capacitor, then a resistor, another resistor here and this visible, yes a capacitor, fine. In this case, we will say here is my loop i 1, here is my loop i 2 and here is my loop i 3. So, my E, L, C 1, C 2, R 1, R 2 quickly write the mesh current equations, write the mesh current equations quickly.

Remember, you have to go around the loops and you have to keep track of the positives and negatives that is what is the most common mistake that you have to make. What, I do will is in order to not to make the errors, I will mark them 0, 1, 2 and 3. And then say the mesh current method will give v 12 plus v 20 plus v 01 is equal to 0, happy there is no problem about it.

Similarly, here it is v 02 plus v 23 plus v 30 equal to 0, similarly here in the loop 3, it is v 1, sorry I will write this way v 13 plus v 32 plus v 21 equal to 0, so these three are the Kirchhoff's voltage law equations. As it trivial, but I did write it this way, because I know, the most students make a problem in signs. So, first write it this way and then substitute these.

We have understood that the right way of choosing the stress the variables would be to say that the inductor current, the current through an inductor is variable and a voltage across capacitor is variable. Why, because of these two, current through the inductor, voltage across the capacitor, these are the minimum variables necessary. So, we will say the current through the inductor and voltage across the capacitor such variables. So, all these currents, I have to be careful about this particular current.

This is what will become a state variable, this is become a state variable, this will become a state variable, but we have to careful about, what we call positive and what we call negative. Suppose, this is call positive, say this direction of the current, say this direction of voltage is positive and say this direction of the voltage is positive, so that is my convention of positivity. Whenever, it takes this value, it will positive, else it negative.

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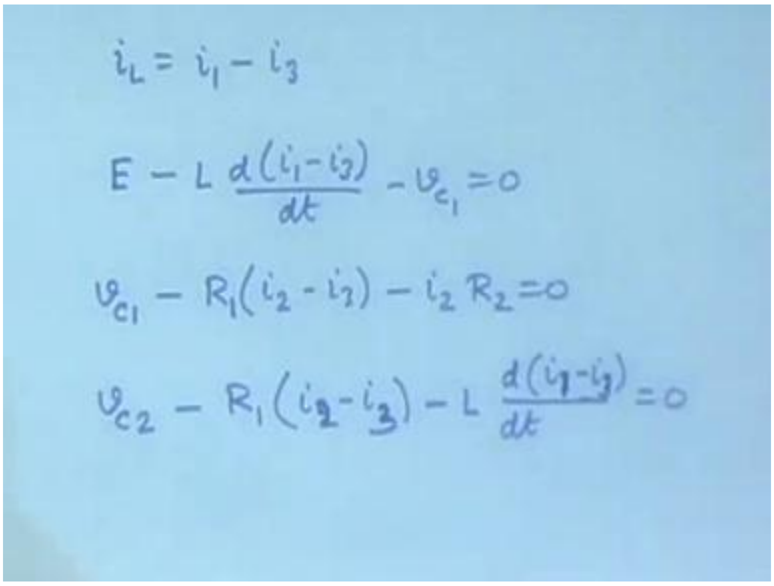

Now, what is i L, i L is the current through the inductor, I L is i 1 minus i 3, fine. So, that is one thing and then the KVL equation becomes write the first KVL equation in terms of that. Now, write it carefully using this convention of positives and negatives, it will become E minus L d i d t. So, minus L d i 1 minus i 3 d t, this is positive and therefore, if go this way it will be negative, so minus v c equal to 0, v c 1, yes.

Now, this negatives and positives are important, I am again insisting on that, because I know in this examination, most of you will make a trouble in that. That is why, I when to carefully this way, so this is one equation this gives a differential equation, but in terms of this, fine second equation becomes and it was this. So, we go up along the v c, so it is v c 1 a minus is drop, so minus R, it will be i 2 minus i 3, i 2 minus i 3, this 1 i R 2 minus i 2 R 2 equal to 0, 1 yes.

Third one, the one at the top, it will give, v c 2, now here I am going anticlockwise is a rise, so I have to go this way minus R 1, it will be i 3 minus i 2, no, i 2 minus i 3 for the correct sense i 3 minus i 2 minus i 3 then minus L. Now, notice the direction this was the direction of i L, am going this way, it is i 3 minus i 1 is flowing this way. So, either you put plus of this or minus of this, so be careful about it.

If I say, if I read write minus here, it is L, what you say d d t of, no, if it is not minus, then i 3 minus i 1, v c 2 up going down, going down plus minus, minus. So, minus it will be a drop, if the current is flowing this way, wait i 1 minus i 3 is the current this way. So, this is this is the equation, once you have obtained this, which are the differential equations, this is the differential equation, this is differential equation, this is not. So, we need to apply something more to that.

So, what any problem.

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No, here is a point, here is a point, here is a point, here is a point and we are going this way up, this way down, this way down, this way down. So, it is i 2 minus i 3 is the current this way, now that is right.

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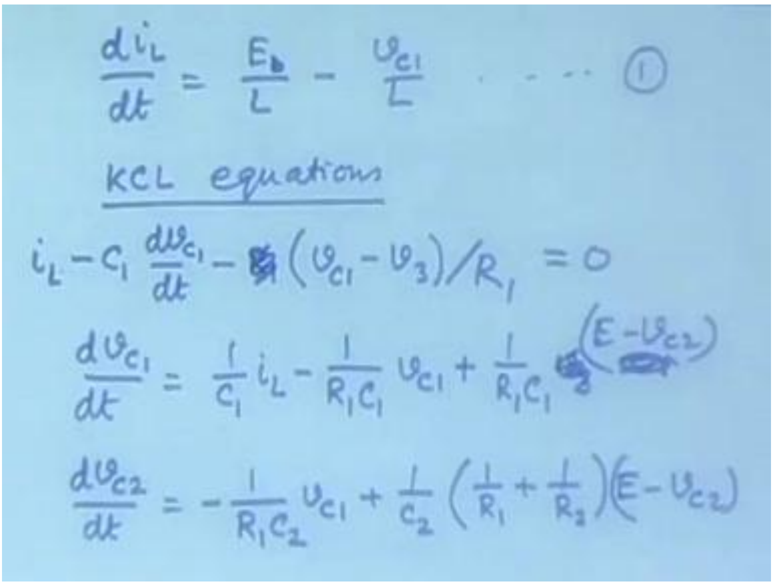

Now, so we have written down the equations, ultimately we need to obtain the differential equation in terms of I L. So, we just substitute it here, we get d i L d t is equal to E by E by L minus v c 1 by L, one equation done, from the first. Now, in order to proceed, we need to write the KVL equations, sorry KCL equations. So, KCL equations are where is the equation, I L here, this current here, this current here, fine let us write it.

In which node should I write, let us write it here at this node, if you write at this node, it will be, can you see that, yes. In this node it will be here it is I L going into it, no I want to write in terms of the dividities. So, it is going into, this is going into and this is going into, so you have to write it that way. So, that is minus C 1 d v c d t and this is again minus R 1 d v c 1. Here, the potential is v c 1 and here the potential is say v 3, this point v c 1 minus v 3, so R 1 v c 1 minus v 3, no I want the current. So, this is the equation in this node.

Now, I need v 3 in order to substitute this, so if can write that easily, but let extract the differential equation first, d v c 1 d t is equal to write the rest of it in the right hand side. It will be 1 by C 1 i L minus 1 by R 1, C 1, v c 1 minus 1 by, wait a minute is this right plus R 1, C 1, v 3 anything else done. Now, can have substituted this point, apply KCL in node 3. What you have apply, write the KCL equation in node 3, it is v 2 minus v 3 by R 1, there is a current here, input.

So, it is this current is minus v 3 by R 2 and the other current is this, this and that, that current would be well this is plus, this is minus, so it is plus C 2 d v c 2 d t equal to 0, from there you can explain. So, just write down this equation in the proper form, so that you extract the differential equation out of it, this in the left hand side and the rest in the right hand side. D v c 2 d t is equal to can you write properly minus R 1, C 2, v c 1 plus 1 by C 2, 1 by R 1 plus 1 by R 2, v 3.

Now, v 3 is what, here also we substitute v 3, here also we substitute v 3, v 3 is what, I need to write down, what is v 3. So, we can extract it from here, v 3, v 23 which I know and v 02, no it be easily extract from this equation, E minus v c 2 is this. So, you can extract from that good, so we just substitute it here, it will be E minus v c 2, here also E minus v c 2.

So, this is how you extract the differential equation, you can see that in keeping the signs correct, we have to really spent a lot of attention. In order to extract the right things, we have to find out, where it that information available, it was spending a great deal of attention. And moreover the problem is that always this assumption that the current through the inductor and voltage across the capacitor will be the minimum number of variables will not work.

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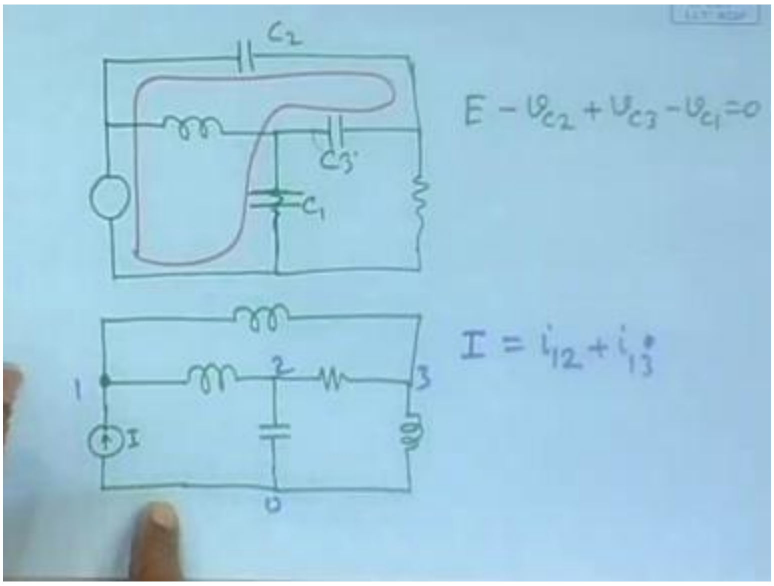

Why, just imagine a circuit, which is something like this, I am just replacing one of the elements in the last circuit, it was like this, sorry it was capacitor and this is a capacitor, this resistor and here is another capacitor. What are you done; I have replaced this resistor by a capacitor. At this time, I can identify a node, an identify a loop like this; it is after a loop, it is not a window, it is not a mesh, but it is a loop.

If you write the case KVL in this loop, what you have, you have E, say this is C 2, this is C 1 and this is C 3. So, E minus v c 2, because this is a plus, this is a minus, then you come here it is plus v c 3, this is plus, this is minus, so minus v c 1 equal to 0. You notice, that the voltage is across the capacitors, these are voltage across capacitors, they no longer independent, why because they there related by this equation.

All of them cannot be independent, because this addition, this is an externally applied voltage, this addition is 0, therefore these 3 cannot be independent and can you see that. If these three cannot be the independent, I cannot assume all the voltages across capacitors as independent state variables. So, here is the problem, notice that let us give another example, say there is a circuit something like this.

I again make a small replacement, here is a voltage source, here it was the inductor and here it was a capacitor that remains the same. Here, it was a resistor that remains the same, here only I replace it by an inductor and here also I replace it by an inductor, suppose I have a circuit like this. See the problem, if I write this node equation, then what we have, suppose this is now a current source I, imagine that the current source.

And in that case, the equation becomes I is equal to I, this 1, 2, 3, 0, i 12 plus i 13, obviously or I minus i 12 minus i 13 is equal to 0, whatever it is, this immediately means that these two inductor currents or not independent variables. So, if you have a circuit like this, then the inductor current no longer remain independent variables. If you have a circuit like this, the capacitor voltages no long remain independent variable.

So, I have identified a few main problems with applying the Kirchoff’s laws in a row way. In a row way, what we have simply taking the mesh current and the node voltage and proceed to write down the differential equations. We have been able to write, but that will true for relatively simple circuits. Even, for a three mesh circuit, it was becoming a bit tired some, so it is not good.

And secondly if you have this kind of arrangements of the circuit, then the basic assumption fails, so we need to have some kind of a systematic way of doing this. In the next class will tackle with that systematic way of doing this.

Thank you very much.