Engineering Mechanics
Prof. Siva Kumar
Department of Civil Engineering
Indian Institute of Technology, Madras
Dynamics of rigid bodies

Now I am going to do one small thing that will help us understand further. When we go back to this, let's deal with translatory motion separately and rotary motion separately. What I am going to describe right now is the rotary motion. In order to understand that what I will do is I will move this reference frame to this point and describe only the rotation. This is just to understand what happens to this vector r A B. Think of it like this. This is the B that we are talking about and let's say this has rotated to become something like this. So point A was here, it has become… this is the final configuration that we are looking at.

What is basically happening is in this rotational motion this vector is rotating about a particular point. Now if I have to understand this let's just go with something that represents this vector as it moves. Let's say I have a point here which is A. Remember as A moves, this line also moves. Supposing the point A comes over here, this also comes over here. If this is B, what is this? This is r of A with respect to B, r of A with respect to B after it has moved.

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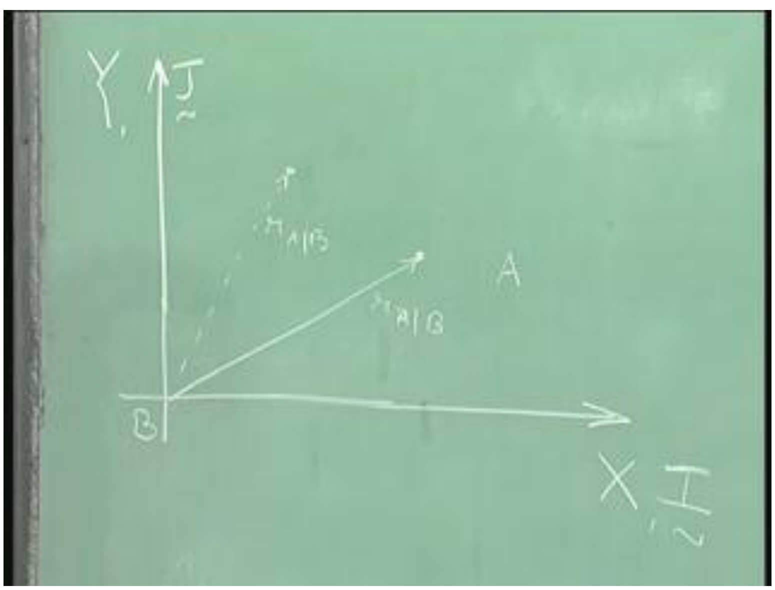

We will represent this by vector e_r which is along this line r A B, r A with respect to B and another which is perpendicular to this which I am going to call as e_{theta}. These two are the unit vectors I am using. Why am I using this unit vector? Very simple. Supposing it rotates this way and reaches this particular point. Let's say this is a new location A prime.

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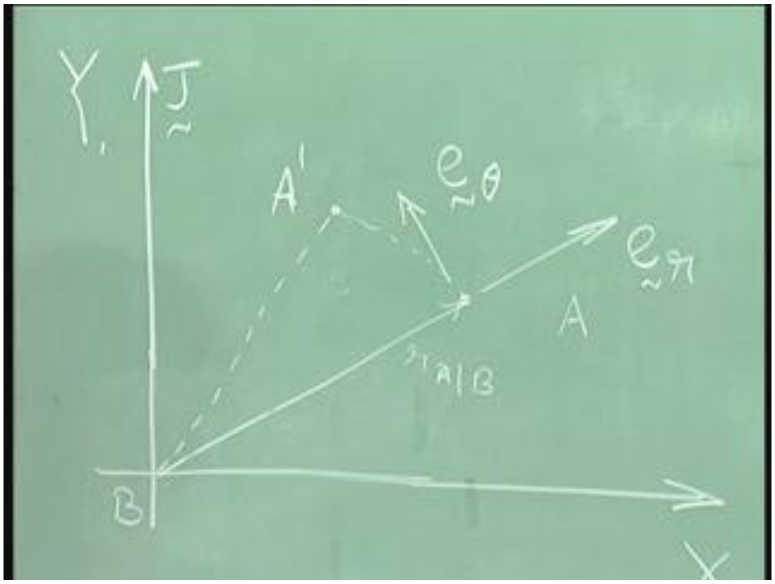

Remember e_r is now rotated this way and e_{theta} also has rotated like this. Let's say it has gone through a rotation of this sort.

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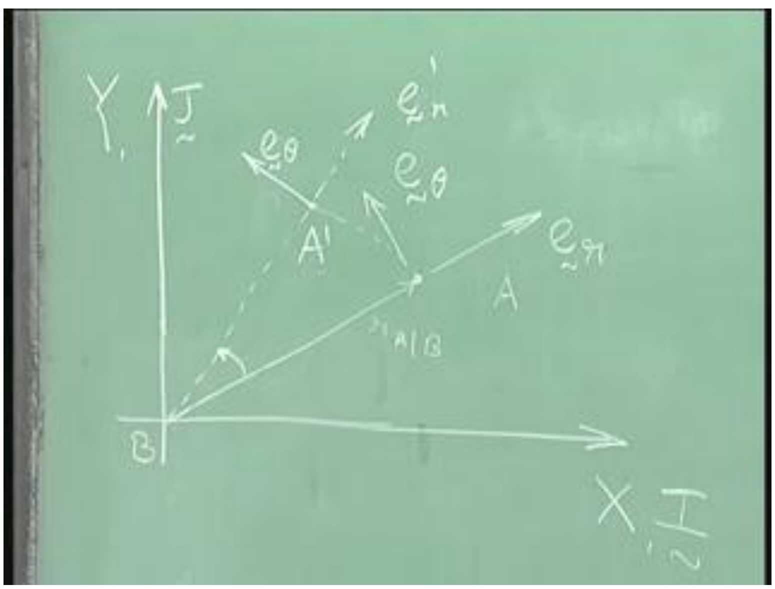

How do I represent r A B? It is nothing but r of A with respect to B is nothing but lets say we will take small r, this is a vector. This is a scalar times the direction is e_r, r A B which is the position vector of A with respect to B can be now written as some value here r which is nothing but the magnitude of the distance between them times e_r. Now I have a representation. The beauty of this representation is remember e_r is now rotating. If I find the time derivative of this rotation of e_r and e_{theta} I will be able to solve this problem.

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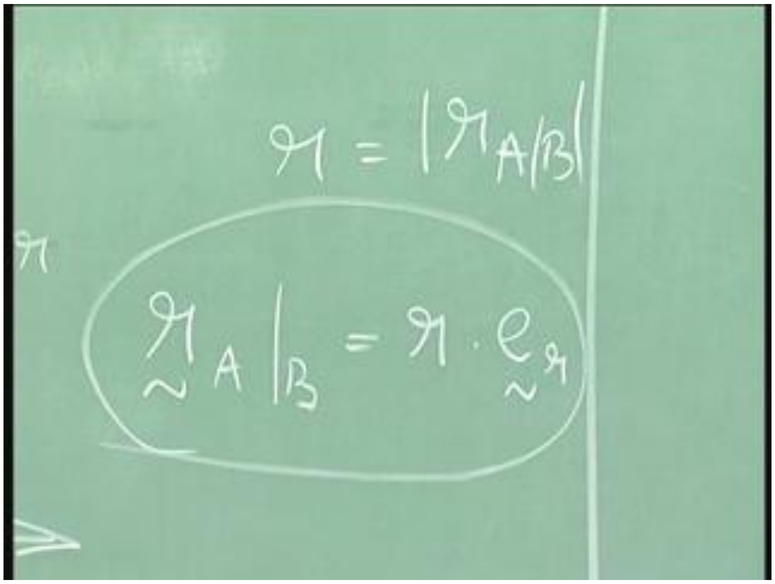

Given this how do I find out r dot of A with respect to B. Most of what we are going to talk about will involve this r dot of A with respect to B. Simple, we will use the simple derivative. The time derivative of this is what is going to give us the derivative of this, time derivative of this. But this is a product of two quantities which are changing with time. Is r changing with time? Not really. e_r is changing with time? The answer is yes. This will be r times, let me just make it general and then use the concept of rigid body. I have just taken the partial derivatives. If you look at this, I have taken the derivative by parts, r dot is zero. Why is r dot zero? Because it's a rigid body, the distance does not change which means this is equal to zero, we have r times e_r dot. I know r that I have specified but what is e_r dot? How do I find out e_r dot? That's the next question we will ask and then find out what is happening. From an understanding of how it is changing here, I should be able to find out what is e_r dot.

Let me just use a separate figure over here in order to understand this. Let's say this is at time t and at time t plus delta t, I will draw one more configuration. This is the vector that you have r of A B, this is e_r. Once it has rotated by lets say a delta theta which is small, it would have reached this particular point. Let's say this is A A prime as we have discussed earlier. We find that e_r has now shifted in its direction. Let me just put it as e_r prime so that we understand this. If I superimpose the directions, I will find that this direction is like this. Do you agree with me? What's the angle between these two? It is delta theta. Let me just zoom this in. I have e_r, I have e_r prime just to give you an idea. The angle that it has gone through this is delta theta. If I have to find out e_r prime, e_r prime is nothing but e_r plus a vector that joins this. Do you agree with me? e_r plus a vector that joins this. So let me just take this vector as t. Do I know the length of this vector?

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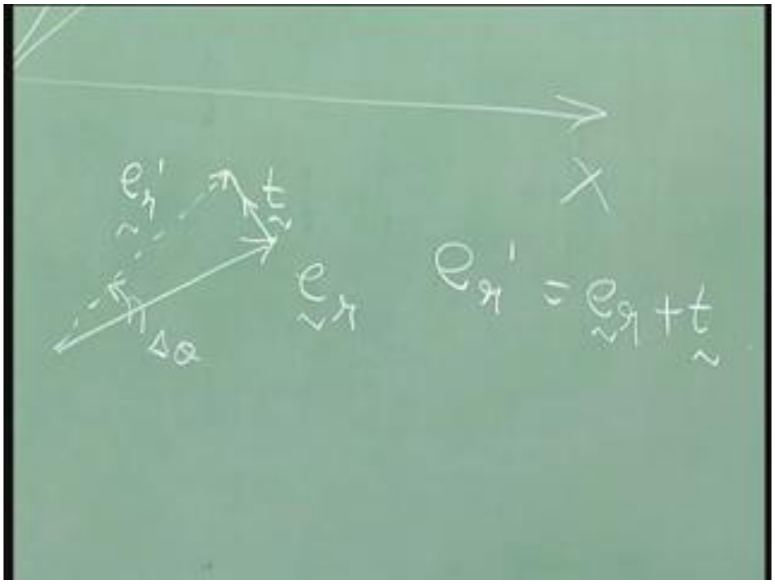

Since this is a small rotation and this length does not change. The length of this vector t is r times delta theta. Its magnitude is r times delta theta. Do you agree with me, where r is the magnitude of r A with respect to B. How about direction? What is the angle between these two? What's the angle between e_r and t? It is tangential or in other words there is a 90 degree angle which means the unit vector along this t should be nothing but as we have mentioned over here e_r and 90 degree e_{theta} which means this will be r times delta theta times e_{theta}.

Do you agree with me? Is this clear? So r times delta theta e_{theta} is the rotation it has undergone over a time delta t. Over a time delta t this is what has happened. To what? To e_r. Let me just write that down here, e_r prime is nothing but e_r plus r delta theta e_{theta}. What we are interested in is how it has changed or in other words e_r dot is what we are looking at. If this is e_r and this is e_r prime which is a change that has occurred over delta t. Then e_r dot is nothing but limit as delta t tends to 0, e_r prime minus e_r divided by delta t. is this okay? This is the changed vector, this is the original vector divided by delta t. What is e_r prime equal to? It is nothing but e_r plus r delta theta e_{theta} or in other words e_r prime minus e_r is equal to r delta theta e_{theta}. What we get here is r delta theta by delta t into e_{theta}. Let me put a limit over here. This is okay?

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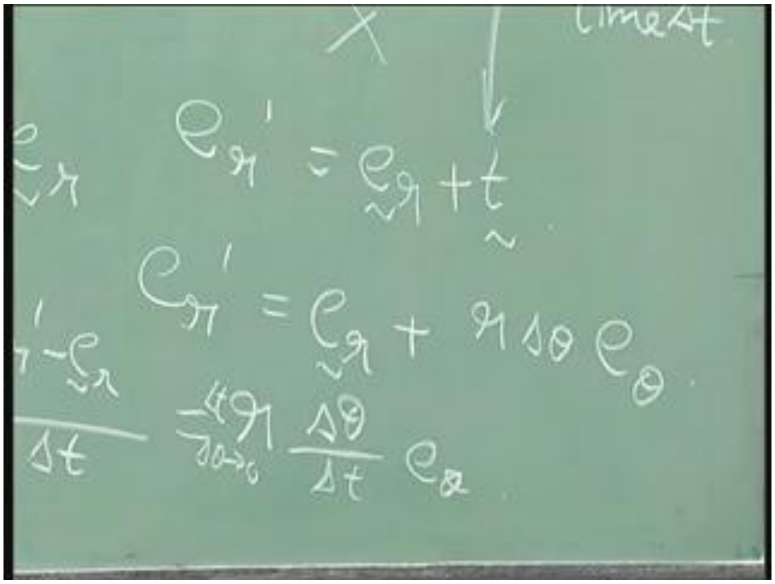

What is limit as delta theta delta t tending to 0? delta theta by delta t it is theta dot, so I have r times theta dot e_{theta} as e_r dot. Let me go back to this and write this to be equal to r times, we have one more r over there. Am I right? What is this distance equal to? Is it r? Let's look at this t, what is the length of this? This is e_r. What is the length of this? What is the length of a unit vector? One, so when I took this r, this r is actually equal to 1. Therefore in this let me just remove r here, remove r here so that we get this r times theta dot e_{theta}. Is this clear? Why is it so? Because e_r dot, rate of change of the vector e_r happens to be equal to theta dot e_{theta}.

We find that this is the way it has moved, so if I zoom in I have this as e_r. What is the length of this particular vector? It is equal to one, it has now changed its direction to something like this. Let me call this as e_r prime. If I take this as let's say alpha vector then I can write e_r prime vector is equal to e_r vector plus alpha vector. e_r plus alpha is equal to e_r prime. What is the length of e_r prime? That is also equal to 1. What is this angle? This is delta theta and this is occurred over a time delta t.

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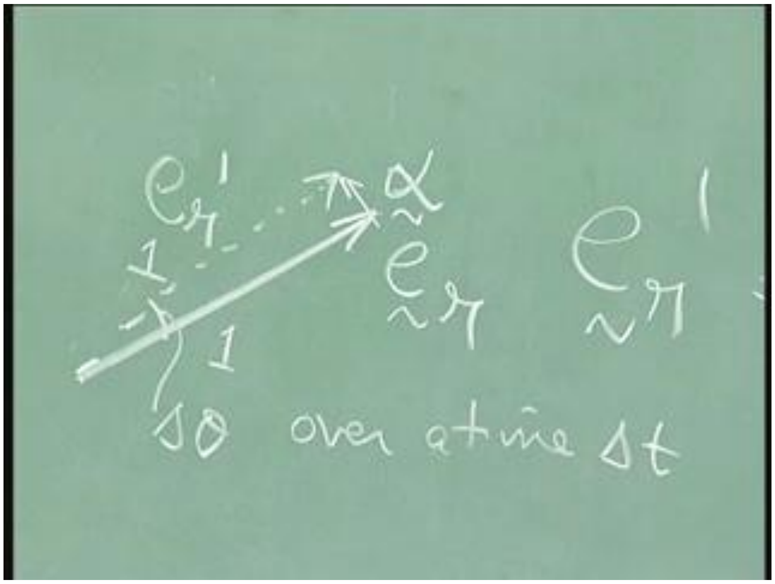

This implies that e_r prime minus e_r is equal to alpha. Let's now focus on alpha. What is the magnitude of alpha? That's not very difficult, since this radius is 1 and it has gone through a delta theta, the magnitude of this should be 1 times delta theta which is delta theta. What is the direction of this? Remember this is along the tangential direction to the sweep that occurs and therefore this is perpendicular to e_r. If you go back to this, what is the direction vector that is perpendicular to e_r that is the unit vector e_{theta}. Therefore alpha can now be written as delta theta times e_{theta}.

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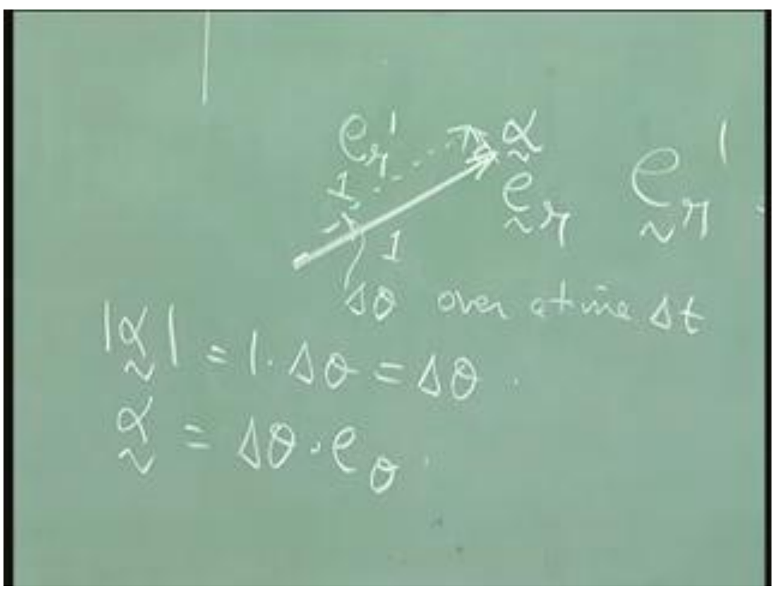

What is the rate of change of this vector e_r? It is nothing but limit as delta t tends to 0. Whatever change has occurred to e_r divided by delta t. The change that has occurred is e_r prime minus e_r which is nothing but alpha. But alpha is equal to delta theta minus e_{theta} and therefore this is delta theta divided by delta t times e_{theta}. So as delta t tends to 0 this becomes theta dot e_{theta}.

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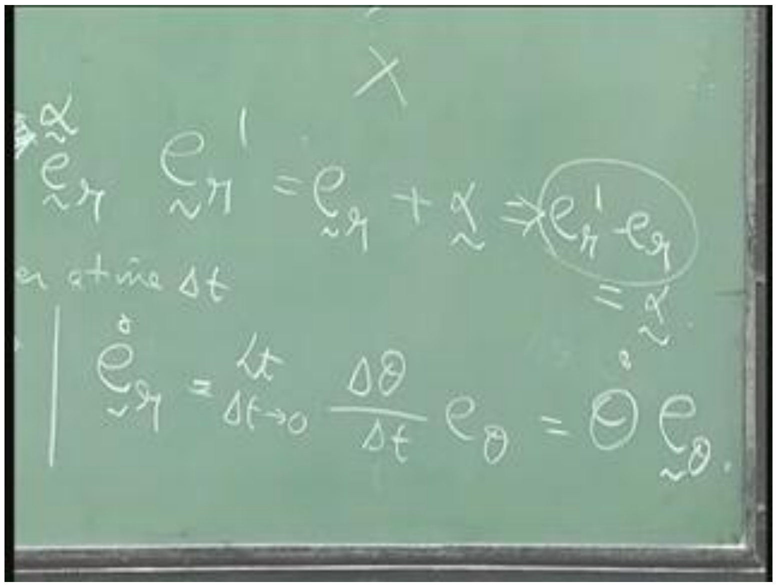

Therefore if I go back to this, what's the derivative of r A with respect to B? It is equal to r times e_r dot but e_r dot is equal to theta dot e_{theta}. If I substitute, I get r theta dot e_{theta}. Just to get an idea r times theta dot is the magnitude of the velocity and the direction is e_{theta}. That's very clear here. This is the direction along which the velocity due to angular motion occurs and r times theta dot is the magnitude of the velocity.