Fundamentals of Transport Processes II
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Lecture - 8
Rate of Deformation Tensor

This is lecture 8 of the course on Fundamentals of Transport Processes. Welcome to you all.

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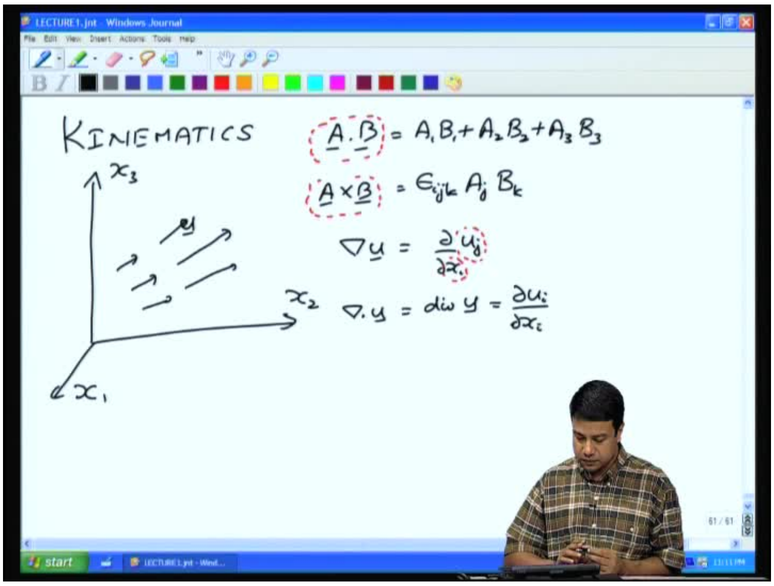

And we will continue what we were doing in the previous lecture, that is on Kinematics. This is the description of fluid flow without reference to the forces that are acting on the fluid. In the case of heat and mass transfer, there was a concentration or a temperature field and there was a flux, that flux can the flux is a vector, it can be directed in any direction in an space, however it has just one direction. In the case of fluid mechanics, we have a velocity filed and that velocity field itself has a direction.

The transfer of momentum there is a direction associated with the momentum itself there is a direction associated with the direction of transfer, and these two things come together. So, in that sense fluid mechanics is a little more complicated than normal heat and mass transfer, and for this reason in the past few lectures we have been conducting preparatory lectures on vectors and tensors their derivatives their integrals.

So, let us just briefly review that before we continue our discussion on kinematics. For any kind of flow say a pipe flow or flow around the particle and so on. We have a three dimensional coordinate system, the velocity in this three dimensional coordinate system is a vector, it is a vector field, it is defined at each point within the flow. It has a direction and the magnitude and both of these can vary both in time as well as in space, the direction and the magnitude of the velocity can vary both in time as well as in space.

We do analyze these of course in a fixed coordinate system with their components in those 3 coordinates, but however, this velocity field itself as I have been trying to emphasize has an identity, which is independent of the underling coordinate system. The flow in a pipe for example, would continue to flow in that same direction regardless of what coordinate system, I use to analyze that flow.

So, you were looking at certain fundamental things about these vectors, which are invariant under coordinate systems. The magnitude and the direction are 2 such fundamental independent quantities, the dot product of 2 vectors is an independent quantity for example, the dot product of 2 vectors A dot B. Even though I can write it, in a coordinate system as A 1 B 1 plus A 2 B 2 plus A 3 B 3, this dot product is independent of the coordinate system, that you use to analyze it depends only up on the vector. Similar is the thing with the cross product, A cross B, which with reference to A coordinate system, I had already shown you, that this can be written as epsilon i j k A j B k resultant is A vector. There is 1 free index, I which gives the direction of the resultant cross product j and k are repeated so they represent dot products.

This once again is independent of the coordinate system that is used, specifically in the context of derivatives of a vectors, we had defined 3 such, one was the gradient gradient of the velocity field, which I can write it as partial by partial x i of velocity j. This quantity has 2 un repeated indices. So, there are 2 directions associated with this quantity, therefore it is a second order tensor, there is one direction associated with the direction of the gradient.

The direction in, which you are measuring variations, this is again, which is the direction of the velocity direction in, which the fluid is moving. So, there are these 2 directions associated with it, this is a second order tensor, which has an identity once again independent of the coordinate system, that you using to analyze up on. There were other derivatives that, we had analyzed one of them was the divergence del dot u, which is the divergence of u in indicial notation, you would write that as partial u i by partial x i, the divergence of A vector.

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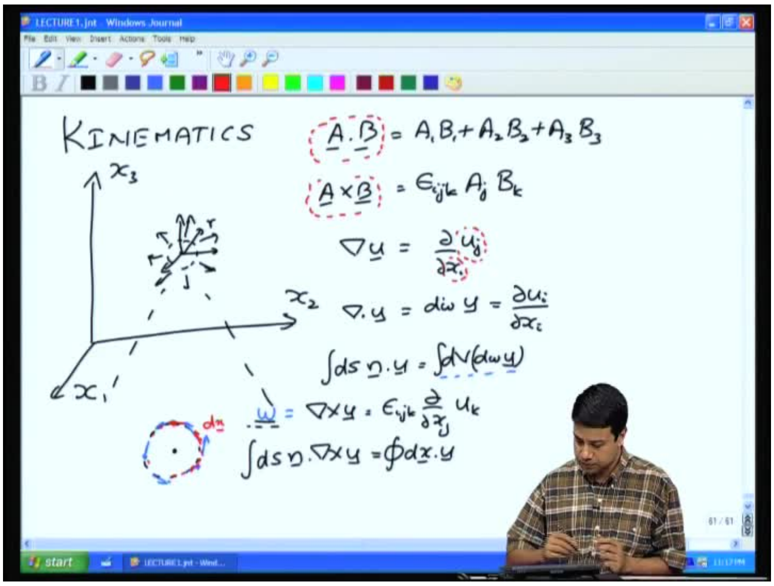

And by an example I had shown, you in the last lecture, that this is non zero at a point only, if I have a velocity component that is diverging radially, outward from that point only, if I have something that is diverging radially outward from this point. In other words, if is it at this particular location and I put in a coordinate system at this particular location and the distance from the origin, that is the distance from the point, which I am sitting is r.

And if the velocity has a component, that is along the vector from the origin then the divergence will be non zero, because you know from the integral theorem for the divergence that integral d s n dot u is equal to integral d v divergence of u integral d s of u dot n. That is let us just expand, this volume out a little bit, this is the point that I am sitting at and I have a small volume around this point, that surface has it is unit normals and then I have the velocity vector, which is in general in some direction.

If, I take that velocity vector dotted with the unit normal integrated over the entire surface, I get the net amount of fluid that is coming out of that surface the volume per unit time, that is coming out of the surface. So, only if there is a net amount of fluid coming out of this surface will I have the divergence of u being non zero within this surface. So, there is a physical implication of the divergence.

And the other vector that, we had defined was the curl del cross u, which is epsilon i j k partial by partial x j of u k, the velocity, that is the curl of the velocity vector. And in the last lecture, I had shown it shown you in a simple example that this curl of this velocity vector. Over some curl over some line in this 3 dimensional space will be non zero, only if there is circulation around this point. We had seen it in the last lecture by taking up velocity vector, which is solid body rotation and we had calculated the integral the the del cross u for that particular solid body rotation with 2 different components of velocity.

And we saw that the verticity, which was equal to the curl of the velocity is non zero, only if there is a circulation around that point. And that verticity is related to the angular velocity of that solid body rotation. There is an integral theorem that, which states that integral over any surface of n dot del cross u, n dot del cross u is just the unit normal dotted with the verticity vector. Note that the verticity vector is equal to the curl of the velocity vector, it is a vector and it is equal to the curl of the velocity vector.

Obviously, it is in a plane perpendicular to the velocity, because when you take the cross product of 2 vectors, you get a resultant that is perpendicular to both of those. So, this verticity vector is in a plane perpendicular to the velocity vector and this integral of the unit normal dotted with del cross u over any surface is equal to integral the line integral of d x dot u.

So, I take this little differential element on the surface, that is d x vector and dotted to the velocity vector at each point and then take the integral over entire contour. That, is equal to n dot del cross u over any surface for, which this contour is the perimeter is equal to n dot del cross u for any surface for, which this contour is the perimeter that was the physical representations. Last class, I had also told you that whenever the curl of a velocity is equal to 0, that velocity can always be represented as the gradient of a potential.

So, that is the velocity potential, so when del cross u is equal to 0, u can be written as the gradient of phi, when the divergence of the velocity is equal to 0, the velocity can always be written as the curl of a vector. When the divergence is 0, the velocity can be written as the curl of a vector That is not simplified things, because velocity also had 3 components and this vector norm has 3 components, significant simplification results when you have 2 dimensional flows with only 2 components u 1 and u 2. In that case the vector whose curl, you take to get the velocity has to be perpendicular to the plane.

So, that we had defined as the stream function vector with direction perpendicular to the plane. So, that basically defines the stream function and I had show you that the along lines of constant stream function, the velocity vector is tangent to lines of constant stream function and the difference in stream function between adjacent lines of constant stream function called stream lines is equal to the net flow through that surface.

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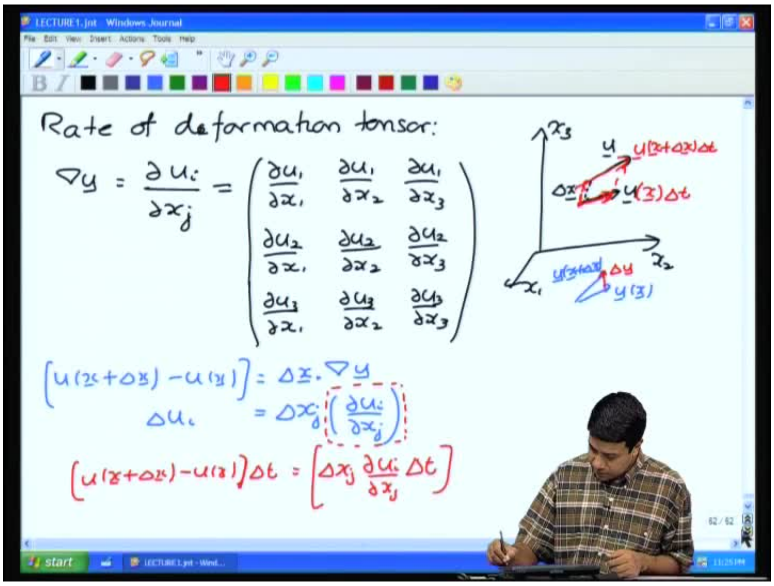

An important concept, which we will discuss, now is the rate of deformation tensor and I had started of the discussion for you in the class. The rate of deformation tensor is the gradient of the velocity, I had written it for you, in the last class as partial u i by partial x j where i is the direction of the velocity j is the direction in, which we are taking the derivative. So, the direction of derivative means that, you go a small distance in that direction finds the change in velocity and divided by the distance travelled.

And as we saw it has different components partial u 1 by partial x 1, it has 9 components. So, first thing is the velocity itself has 3 components at any position. This is the velocity, it has 3 components at any position and then I can go a small distance delta x, this distance travelled can also be in 1 of 3 directions and then I find out what is the velocity at this other position. Note that, I am travelling in in space while keeping time a constant, so at a given instant, I am travelling a small distance in space and seeing, how the velocity changes.

So, this is the velocity vector, I will just write it separately, this is u at x, the velocity vector that, I get at the new position u at x plus delta x is in general slightly different than that at u of x. So, I will get u at x plus delta x minus u of x, which is the difference in velocity. So, the difference in velocity is this 1 delta u, that is equal to delta x dotted with the gradient of u. So, this I can write it for example, as delta x i, so I should write be careful here, delta x j times partial u i by partial x j.

So, there is a difference in velocity, that is delta u i is equal to delta x j times partial u i with respect to x j right hand side, 1 repeated index dot product free index is i is the same as the one on the left hand side. So, this thing this second order tensor here, contains all the information about how the velocity is changing as I move a small distance away from a given location in space.

Note that this partial u i by partial x j is a function of position, it varies as you go to different points in the velocity field, but it is not a function of the distance travelled from that point from the definition of the gradient. The gradient definition basically states, if I start at some particular point and move a small distance delta x from that point, the difference in a quantity whether it is a scalar or a vector is going to be equal to delta x dotted with the gradient of that quantity provided delta x is sufficiently small in the limit as the length of delta x goes to 0.

So, therefore, this thing, this gradient of the velocity is something that is defined at each point in space, it is dependent up on position it could depend up on time, because velocity itself is changing with time, it does not depend up on delta x. So, it is it is a it is a function of the velocity field itself. So, if I am sitting at some particular location, I can always find out how much the velocity is changing as I go to nearby locations, it is in a similar manner to the temperature gradient, it gives me the direction in, which the temperature is changing as I go s it at one location and go to nearby locations, it tells you how the temperature is changing. The other thing it is telling, you is that if 2 nearby points, if I have 2 nearby points here, if I have 2 nearby points here and I wait sometime delta t, if I wait sometime delta t right.

So, if I have 2 nearby points and if I wait a time delta t, this distance that this point travels, the distance that this point travels is going to be equal to u at the location x times delta t e. So, the distance that the first point travels point at the location x travels is going to be u at x times delta t and the second point here is going to travel some other distance. The second point here is going to travel the distance u at x plus delta x times delta t. So, therefore, the change in this displacement vector between these 2 locations the change in the displacement vector between these 2 locations is going to be equal to u at x plus delta x minus u at x times delta t, which is equal to grad u delta x j partial u i by partial x j times delta t.

So, this line element delta x j has changed by an amount equal to partial u i by partial x j times delta t when, you have travelled this distance. So, this basically tells, you how the distance between 2 nearby points within the fluid is changing as time is evolving and since all stresses, which are exerted on the fluid depend only up on the rates at, which nearby points are are changing the distance. It is this gradient of the velocity vector, which basically written as the stress in a fluid that is the reason this is important. Now, this is this velocity gradient is second order tensor, it has 9 components and you can of course, write down the components in in different directions. However there is a fundamental way in, which you can you can ah classify deformation on the basis of how this this rate of deformation tensor, what are the fundamental invariant components of this rate of deformation tensor. Let me just explain that a little bit now.

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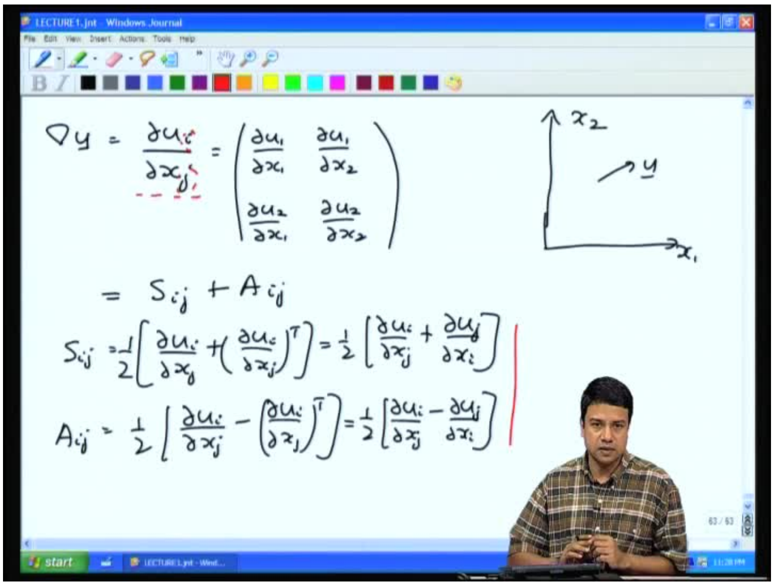

So, in the previous lecture, we had started that so my is equal to partial u i by partial x j, it had it has a total of 9 components. Further purposes of the present discussion, I will restrict attention to A 2 dimensional velocity field, some velocity vector u, in this 2 dimensional velocity field. We will see later how this is extended to 3 dimensions, but it is easier to visualize, if I do it in 2 dimensions.

So, in 2 dimensions there are 2 coordinates and 2 velocities and therefore, this gradient of the rate of the rate of deformation tensor has only 4 components. Now, this any matrix of this kind these are always square matrices, because in 2 dimensions, it is 2 by 2, in 3 dimensions, it is 3 by 3, so they are always square matrices. Any matrix can be written as the sum of a symmetric and an anti-symmetric matrix.

So, this can always be written as the sum of a symmetric matrix and an anti-symmetric matrix where, S i j is equal to partial u i by partial x j plus whole transpose. There is half the sum of the matrix plus it is transpose, the sum of the matrix plus it is transpose and A i j is equal to half of partial u i by partial x j minus partial u i by partial x j, the whole transpose what does it mean to take the transpose.

So, in this matrix, I was representing the the row indices this I was representing the row indices i is 1 is the first row and 2 is the second row, j represents the column index when you take the transpose, you interchange the row and the column indices. So, in the transpose of the matrix, if the original matrix had an element m 1 2 in the transport that element would go into the location and m 2 1.

So, I am interchanging the row and the column, since high represents the row index and j represents the column index, the transpose just corresponds to interchanging the row and the column that is the second one, I interchange, the row and the column. So, I get partial u j by partial x j, similarly for the anti symmetric tensor and you can easily verify that, if I take these 2 and add them up, if I take these 2 and add them up the symmetric and anti symmetric, I get back to this tensor. So, I have separated out the the rate of deformation tensor in the simple 2 dimensional system into a symmetric and anti-symmetric part, I can further separate out the symmetric part.

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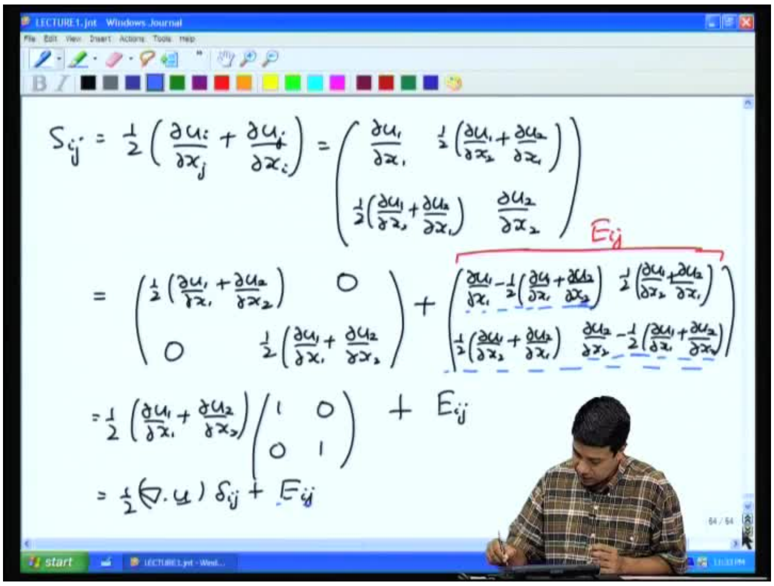

So, the symmetric part, if I write it out S i j is equal to half partial u i by partial x j plus partial u j by partial x i. So, this has components, I am sorry, partial u 1 by partial x 2. So, this is the symmetric matrix and I can further separate out into 2 parts 1 is what is called an isotropic matrix, the isotropic matrix has as it is diagonal elements the trace of this matrix divided by 2. So, this I can write it as half, so that is the first part, you can easily identify this this also can be written as half of partial u 1 by partial x 1 plus partial u 2 by partial x 2, this scalar quantity times, the identity matrix. So, that is the first part and there is a second part, which is basically the original symmetric matrix minus the 1 that, I have already have here.

So, this will be partial u 1 by partial x 1 minus half and this this partial u 2 by partial x 2, I made a mistake here. So, this is the matrix, that is left over once, I have taken out this half of this, I have taken out this isotropic matrix and this thing is what is called the symmetric traceless matrix is called a symmetric traceless matrix. And this first one is an isotropic matrix, the first is an isotropic matrix. So, this is equal to half, we can very easily see that partial u 1 by partial x 1 plus partial u 2 by partial x 2 is the divergence of the velocity in 2 dimensions. So, this is the divergence of the velocity in 2 dimensions. So, this is divergence of velocity times the identity matrix, the identity matrix is delta i j and the second is the symmetric traceless matrix.

This matrix, which I have written out in in long from here, it has the property that the trace is equal to 0, the trace is the sum of the diagonal elements here and you can easily verify that, if I add up the 2 diagonal elements here, I will end up getting 0, if I add up the 2 diagonal elements here, I will get end up getting 0. So, this is a traceless matrix and the second matrix is an identity matrix.

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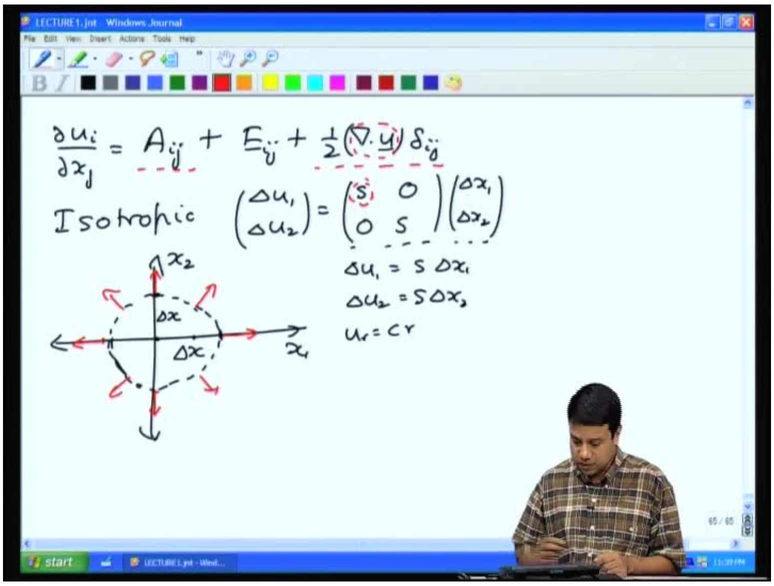

So, therefore, I have separated out my rate of deformation tensor into 3 parts, one is the anti symmetrics plus the symmetric traceless plus an isotropic matrix, which in 2 dimensions is half divergence of u into delta i j. So, one can now ask the question, what types of deformation do each of these represent. So, if you recall, we had a in in 2 dimensions, we have a 2 by 2, that is 4 4 terms in the matrix, the anti-symmetric matrix has only one independent element, because the diagonals are 0 and the off diagonals are equal to negative of each other. The isotropic matrix has only one element the symmetric traceless matrix in addition has in general has 2 independent elements the sum of the diagonal elements has to be equal to 0, for the symmetric traceless matrix. So, let us look at the type of deformation that each of these represents.

So, first let us look at the isotropic term, therefore if I am sitting at 1 particular location note then for the the difference in velocity, this is x 1 x 2, I have delta u 1 delta u 2, the 2 components of the velocity is equal to this matrix times the distance, I have travelled. So, the difference in velocity between 2 locations is equal to the gradient of the velocity times the displacement travelled. For an isotropic matrix, I have only 2 diagonal elements, which are non zero and they are both equal the off diagonal elements for the isotropic matrix are 0. So let me just for simplicity call the diagonal elements of this as S and the off diagonal elements are 0. So, this is the kind of deformation, I will get difference in velocity when I travel a small distance delta x 1 delta x 2 in the 2 directions for an isotropic matrix. So, you can easily simplify this to get the velocity difference delta u 1 is equal to S delta x 1 delta u 2 is equal to S times delta x 2.

So, if S travel a small distance delta x 1 in the plus x 1 direction, it travel a small distance delta x 1 and the plus x 1 direction. Then my velocity vector delta u 1 is equal to S into delta x let us call this delta x, let I have a small distance delta x in the plus x 1 direction, delta u 1 is equal to S times delta x delta u 2 is equal to 0, because I have not travelled in the x 2 direction. So, I get a velocity vector that looks like this at this location, if I travel a small distance in the plus, if I travel a small distance delta x in the plus x 2 direction then delta u 1 is equal to 0, delta u 2 is equal to S times delta x delta u 1 is equal to 0 delta u 2 is equal to S times delta x that is the velocity vector. This is the relative velocity vector of this point relative to the origin, I am always taking the difference in velocity delta u 1 and delta u 2 between a near by point and the origin.

Similarly, if I travel in the minus x 1 direction delta u 1 is equal to minus S delta x, because delta x is now negative delta x 1 is equal to minus delta x therefore, delta u 1 will be in the minus delta x 1 direction. Similarly, over here. In fact, this was exactly the velocity profile that, we had solved in the previous lecture, if you recall we had said that u r is equal to some constant times r and u phi is equal to 0, so if you. So, these are at at the along the coordinate axis, but if I draw a sphere around this and I calculate the velocity at various points, you find that the velocity due to this isotropic part is radially outward. So, the velocity profile due to this isotropic part of a rate of the deformation tensor is always radially outward no surprise, because this coefficient here S was equal to the divergence of the velocity field.

And I told you, that the divergence is non zero, only if there is a radial outward or inward flow, there is an radial outward flow then the divergence is positive fluid ah fluid material points are moving outward relative to the center. If it is negative, they are moving inward towards the center, there is a net volume of fluid that is coming out and that can come out only, if there is a source at the origin. So, this isotropic part is related to the outward or inward flow radially away from the origin or towards the origin. So, this is just one part of the rate of deformation tensor, the isotropic part. Next let us look at the anti-symmetric part. Next let us look at the anti-symmetric part.

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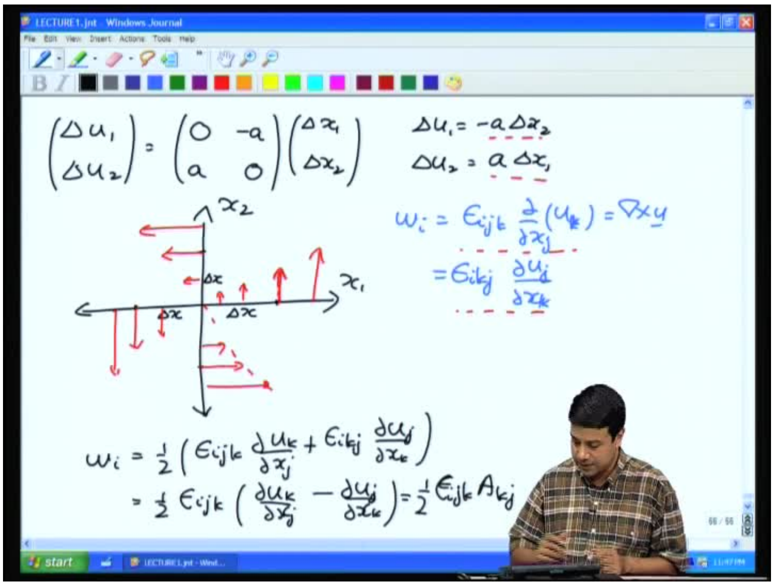

So, for the anti-symmetric part delta u 1 delta u 2 will be equal to some deformation tensor times delta x 1 delta x 2, this tensor is anti-symmetric. So, in general, I can write this as 0 minus a a and 0. So, this is an anti-symmetric tensor times delta x 1 delta x 2 gives me the relative velocity with, which nearby points in the fluid are moving a coordinate [sys/system] system once again. And, if I move a small distance in the plus therefore, this gives me the velocity delta u 1 is equal to minus a delta x 2 delta u 2 is equal to a delta x 1. So, delta u 1 is minus a delta x 2 delta u 2 is equal to a times delta x 1, if I move a small distance delta x in the plus x 1 direction, delta u 1 is equal to 0 and delta u 2 is equal to plus a times delta x. So, delta u 2 is in this direction.

So, the velocity vector at this location will be in the plus x 2 direction, if I go a small distance delta x in the plus x 2 direction then I have delta u 1 is equal to minus a delta x. So, I moved a distance delta x that is means that delta u 1 is equal to minus a times delta x delta u 2 is equal to 0 delta u 1 is equal to minus a delta x and delta u 2 is equal to 0, if I move a small distance minus delta x, if I move a small distance delta x in the minus x 1 direction to this location.

This is once again delta x i moved in the minus x 1 direction, delta u 1 is equal to 0 delta u 2 is equal to minus a, because the distance delta x 1 is minus delta x. So, I get a velocity vector like this and if I go in the minus x 2 direction equal distance delta x, if I go in the minus x 2 direction equal distance delta x then delta u 1 is equal to plus a, because I have gone delta x 2 as minus delta x. So, delta u 1 will be plus a times delta x and I get a velocity vector like this. So, you can see the anti-symmetric part of the rate of deformation tensor corresponds to the rotational flow around this. It is in fact, a solid body rotation, because the velocity the relative velocity of nearby points increases linearly with distance from the origin the relative velocity of nearby points increases linearly with distance from the origin. So, at this point, I will get a velocity like this here and here it will be still larger.

And similarly, in all of these directions, it increases linearly from the origin that means that this represents a solid body rotation around the origin, no surprise, we had associated verticity with the angular velocity of the the the fluid around a local around point. So, this anti symmetric part of the rate of deformation tensor represents the local rotation rotation of the motion of the fluid around this point. Isotropic part is the radially outward flow, anti-symmetric part is the local rotation in fact, you can relate the anti-symmetric part of the rate of deformation tensor directly to the verticity. We know that the verticity omega i is equal to epsilon i j k d by d x j of u k del cross u. Now, I can use a symmetry transform to write this also as rather than, I have 2 repeated indices here j and k, they are repeated, they are summed over 1 to 3.

So, I can very easily inter change j and k and the result will be the same, because j and k are just indices that are summed over, they are dummy indices. So, I can inter change the indices to get exactly the same result. So, this is also equal to epsilon i k j partial u j by partial x k, because I can inter change these 2 and the result remains the same. So, omega i is equal to this 1, it is also equal to this 1. So, I can very easily write it as half the sum of those 2, if some quantity is equal to a, it is also equal to b. It is also equal to half of a plus b therefore, I can also write omega i as half of epsilon i j k partial u k by partial x j plus epsilon i k j partial u j by partial x k. However, epsilon i k j is equal to minus of epsilon i j k, because when I inter change 2 indices, in this anti symmetric tensor, I get the negative of the result.

So, you can also write this as half epsilon i j k into partial u k by partial x j minus partial u j by partial x k. Now, I have defined for you the anti-symmetric tensor A i j as partial u i by partial x j minus partial u j by partial x i. Therefore, the verticity is related only to the curl acting on, I am sorry, the the cross product acting on the 2 components of the anti-symmetric tensor epsilon i j k times k k j. So, the verticity can be related directly to the anti-symmetric part of the rate of deformation tensor.

Verticity does not depend on any other parts, it does not depend up on the isotropic part, because the isotropic part, it will be 0 for a symmetric tensor, if I take the vector minus it is transfers I will get 0. So, the only one that is left is the anti-symmetric part of the rate of deformation tensor. So, the anti-symmetric part of the rate of deformation tensor represents rotational flow, the axis of rotation is perpendicular to the plane of rotation. The plane of rotation is given by this anti symmetric part of the rate of deformation tensor, therefore the axis of the rotation is perpendicular to the plane in, which this anti symmetric part rests.

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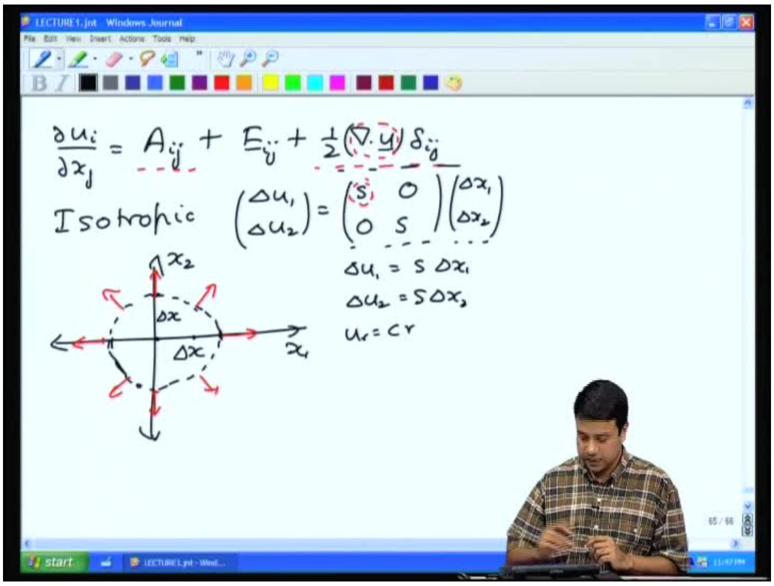

That is the second part first isotropic, which we just done here radially outward motion out of these 3. This is the isotropic part it represents radially outward or inward flow is proportional to the divergence of the velocity.

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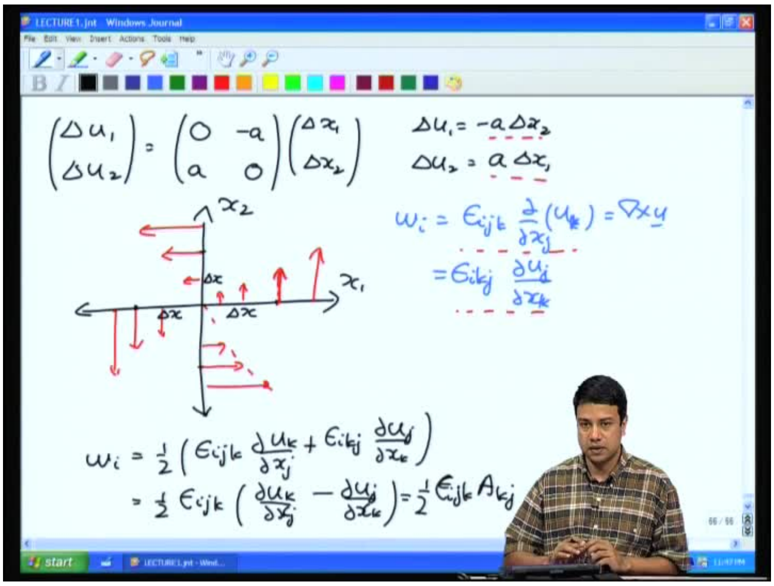

Second is the anti-symmetric part of the rate of deformation tensor, it represents a rotations solid body rotation about the center point, about the origin of this coordinate system tells, you how nearby points are rotating around the location at which is sitting. So, later to the verticity, the curl of the velocity at that point the anti-symmetric part of the rate of deformation tensor can be written directly in terms of the curl of the velocity.

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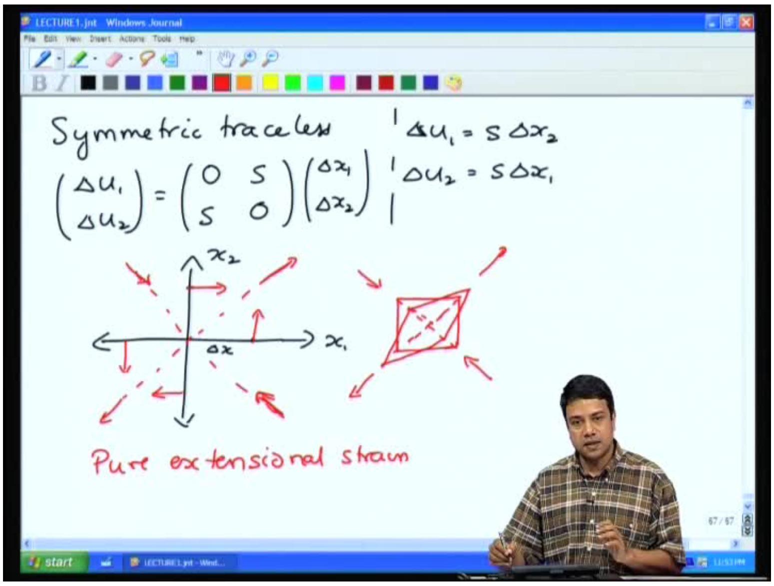

The third part is the symmetric traceless symmetric traceless part for this 1 delta u 1 delta u 2, this is symmetric and it is traceless. So, there are different ways that 1 can write this but you get the same type of deformation regardless of, which way you write it.

So, this is equal to 0, a symmetric tensor, so it is a symmetric tensor, I have said of course, the sum of the diagonals has to be equal to 0. I have said the diagonal themselves individually equal to 0, but that is not there is no loss of generality, there because I can always rotate my coordinate system in such a way that the diagonals become 0, I will discuss that a little later.

So, let us see what kind of deformation that represents. So, in this case delta u 1 is equal to S times delta x 2 delta u 2 is equal to S times delta x 1. So, this is x 1 x 2, if I go a small distance delta x, in the x 1 direction delta u 1 is equal to 0, because delta u 2 is equal to 0, I am sorry, delta x 2 is equal to 0, therefore delta u 1 is equal to 0 delta u 2 is equal to plus S times delta x 2. This something that goes like this, if I go a small distance in the x 2 direction, if I go a small distance in the x 2 direction then delta u 1 is equal to plus S times delta x 2.

And u delta u 2 is equal to 0 delta u 1 is equal to plus S times delta x 2 and delta u 2 is equal to 0 when I go in the minus x 1 and x 2 directions, the directions of these arrows are reversed that is quite easy, because I go minus delta x 1 and delta u 1 is equal to 0 delta u 2 is equal to minus S times delta x 2 and this 1. You can do it at various other intermediate directions. So, in this direction along the 45 degrees in this direction, you will have outward here and along the 45 degrees in this direction will have inward here. So, this is called pure extensional strain, this is called pure extensional strain, so in this particular case, you can see that, if I had the volume of fluid that initially, look like this, if I had a volume of fluid that initially look like this. It is getting stretched along the plus 45 degrees when it is getting compressed along the minus 45 degrees here.

So, after sometime what it will look like is something like this, the axis remain the same beacuse I have taken out the rotational part. So, the axis do not rotate. So, the axis remain the same gets stretched along 1 direction, compressed along the other direction in such a way that the axes are not rotated. And Secondly, there is no expansion in volume, beacuse the expansion in volume was related purely to the isotropic part. So, there is no expansion in volume here, there is no rotation this is pure extensional strain this is pure extensional strain, there is no extension and there is no rotation in this case. In a fluid, we will see a little later that a solid body rotation cannot generate internal stresses, because when you take an object and rotate it, if it is a solid body rotation the distance between nearby points has not changed at all. There is no extension or compression of distance between nearby points.

So, the stresses that are generated cannot be due to the solid body rotation, so the anti-symmetric part of the rate of deformation tensor cannot generate a stress. If the fluid is incompressible the isotropic part also is equal to 0, because there cannot be volumetric expansion or compression, if the fluid is incompressible. The only part that will generate a stress is this extensional strain, the 1 that does not cause any rotation, the part. Of course, the the total fluid velocity field, it is the superposition of all of these 3, I have set it out into 3 parts 1 of, which is radially outward or inward the isotropic part, the other is a rotating part that is the anti-symmetric part of the rate of deformation tensor. And this is a third part and it is only this part that can generate a stress, because it is this one that effects the relative position between nearby points in the fluid. So, this was for 2 dimensions exactly the same can be extended to 3 dimensions.

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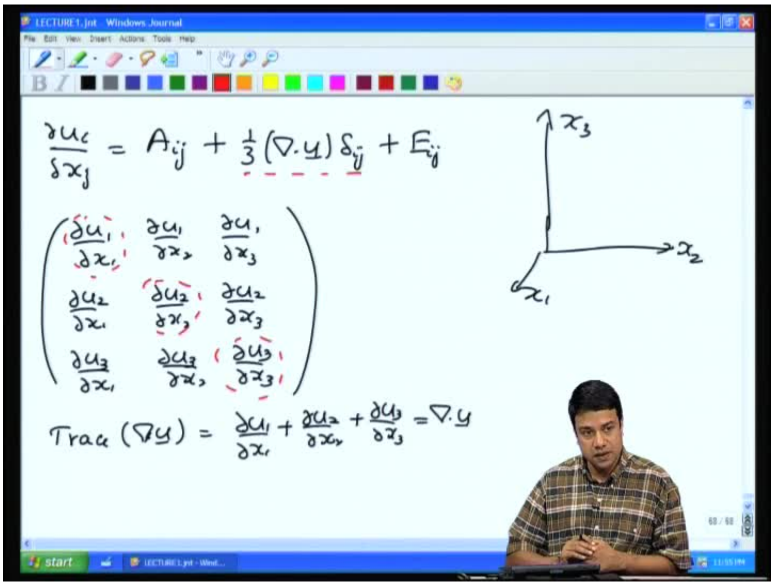

So, let me just explain to you how that is done. So, if I had a general 3 dimensional velocity field, partial u i by partial x j. This I can divide into 2 into 3 parts 1 is A i j plus the isotropic part is actually, equal to 1 by 3 in 2 dimensions, it was 1 by 2, but in 3 dimensions, it is 1 by 3 del dot u times the isotropic tensor plus E i j. E i j is an anti-symmetric tensor and E i j is a symmetric traceless tensor. So, for example, if I had this this was my rate of deformation tensor the trace of this in 3 dimensions is equal to partial u 1 by partial x 1 plus partial u 2 by partial x 2 plus partial u 3 by partial x 3, which is the divergence of the velocity in 3 dimensions that is the trace of the tensor.

And in order to make my my my tensor traceless what I need to do is to subtract out 1 third of this from each of these diagonal elements in order to make it traceless, I need to subtract out 1 third of it from each of these elements. And that is the reason I have this 1 third here, instead of half, because I have subtract out 1 third, I get partial u 1 by partial x 1 plus partial u 2 by partial x 2 plus partial u 3 by partial x 3 and I subtracted out 1 third from each of them. So, it trials up to 1 and I will get something that is traceless.

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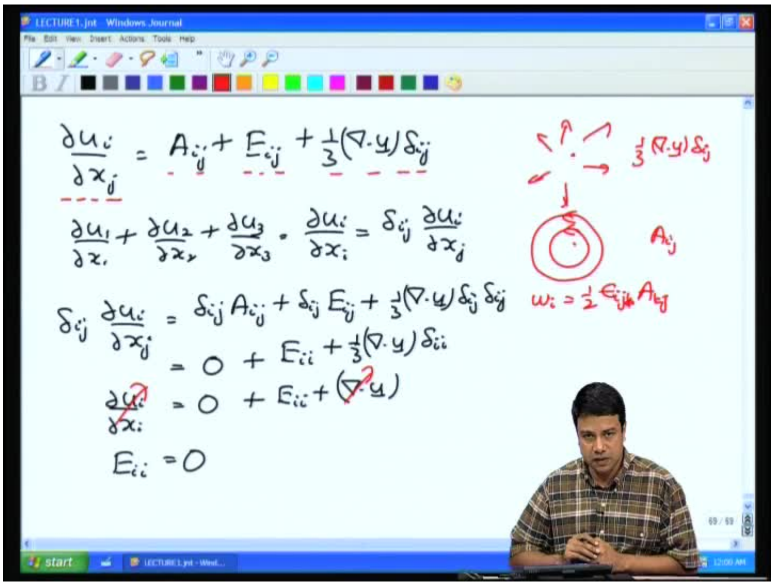

The easier way to do see see it from tensor notations, if I have if i have partial u i by partial x j is equal to A i j plus E i j plus 1 by 3 del dot u delta i j, what is the trace of this tensor. The trace of this tensor is the sum of the diagonal elements the sum of the diagonal elements is given by partial u 1 by partial x 1 plus partial u 2 by partial x 2 plus this is equal to partial u i by partial x i 1 repeated index summed over i is equal to 1 to 3 no unit vectors, so that is the trace.

This also I can write it as delta i j times partial u i by partial x j delta i j times partial u i by partial x j. So, to get the trace of this tensor u dotted with delta i j 2 repeated indices, you end up to the scalar. So, let us see what, you get here delta i j times partial u i by partial x j is equal to delta i j A i j plus delta i j E i j plus 1 third del dot u delta i j delta i j delta i j times A i j, A is anti-symmetric, delta is symmetric. If I multiply a symmetric and anti-symmetric tensor, if I take the double dot product, you will get 0, if you take the double dot product of a symmetric tensor with an anti-symmetric tensor, we will get 0, it is quite easy to see, you can just do expand it out and work it out. So, this becomes 0 plus this delta i j times E i j, I just add up the diagonal parts.

So, when I have a summation delta i j times E i j, this summation i is equal to 1 2 3 j is equal to 1 2 3 of delta i j times E i j that means that in 1 of the summations, I could have replaced i by j, because you get a non-zero result only when i is equal to j. So, this becomes E i i plus 1 third del dot u delta i j times, delta i j is just delta i i beacuse it is non-zero only when i is equal to j what is delta i i delta i i is delta 1 1 plus delta 2 2 plus delta 3 3 1 plus 1 plus 1 that is 3. So, this becomes equal to 0 plus E i i plus del dot u.

On the left hand side also I have delta i j times partial u i by partial x j, which is basically equal to partial u i by partial x i, left hand side, I have del dot u in the right hand side, i have the trace of e plus del dot u. So, these 2 will cancel out to give me the trace of this is equal to 0, the trace of this is equal to 0. So, this is symmetric traceless, this is anti-symmetric and this is the isotropic part, the sum of the diagonals is equal to the divergence of the velocity times one third of the identity tensor in 3 dimensions.

So, this part here of course, represents radially outward flow, it is not a 2 dimensional outward flow, but rather a 3 dimensional from A point, you have fluid coming out in all 3 directions. So, that is the isotropic part of the rate of deformation tensor the symmetric traceless part represents a solid body rotation, the symmetric traceless part, I am sorry, the the anti-symmetric part represents a solid body rotation, so the second part. So, this is one third del dot u delta i j.

This A i j represents a solid body rotation around this point, it represents a solid body rotation around this point, we saw that the verticity vector omega i is equal to half epsilon i j k A k j. The direction of the verticity is the direction around, which there is a rotation. So, the direction of the verticity is basically, equal to half of epsilon double dotted with a anti symmetric part of the rate of deformation tensor. So, if this rate of deformation tensor is in some particular plane, the verticity is perpendicular to that and this A j k gives me, the verticity direction solid body rotation about that and this third part is a symmetric traceless tensor. So, let us just look at that.

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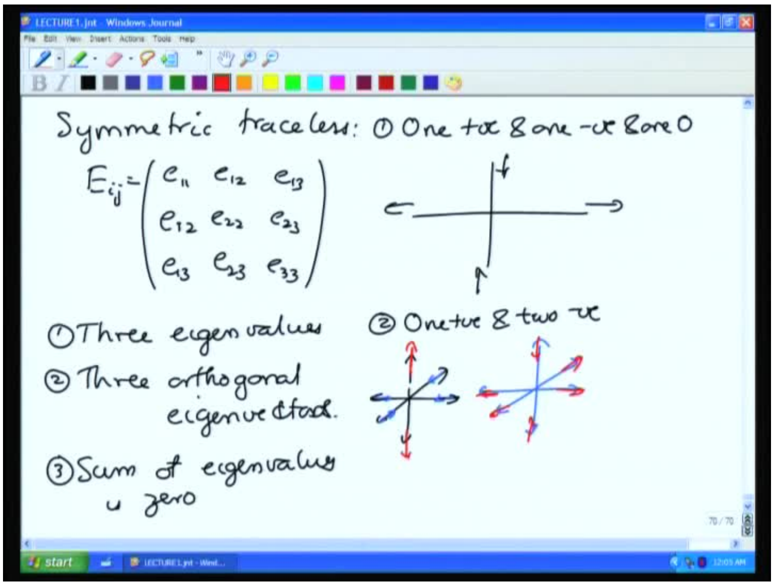

Symmetric traceless E i j, I can write it as e 1 2 e 1 2 e 1 3 and this is symmetric therefore, e 1 2 is equal to e 2 1 and e 1 3 is equal to e 3 1, it is important to note that symmetric tensor the sum of the diagonals is equal to 0. So, this symmetric tensor has 3 Eigen values, it is a third order tensor. So, it has three Eigen values 3 Eigen values, which are real. So, it is a symmetric tensor, you know that, if this tensors equal in transpose it is Eigen values are real and 3 orthogonal Eigen vectors that is 3 orthogonal Eigen vectors in addition. Since it is traceless, you know that the sum of the Eigen values has to be equal to the sum of the diagonal elements of the matrix. Since it is traceless the sum of Eigen values is 0, therefore, you can either 1 positive and 2 negative Eigen values 2 positive and 1 negative or you can have 1 0 and 1 positive and 1 negative Eigen values.

If you have 1 positive and 1 negative Eigen values that means, that there is extension along 1 direction and compression along 1 direction. So, if I have 1 positive and 1 negative and 1 0. If I have 1 positive 1 negative and 1 0 have extension along 1 direction compression along the other direction and there is no deformation in the third direction extension along the direction on, which the Eigen value is positive. You know that the Eigen vectors have to be orthogonal. So, the direction corresponding to 1 Eigen value has to be perpendicular to the direction corresponding to the other. So, this is plane extension extension in one direction compression in the other direction and there is no flow in the other direction, third direction. The second one is when there is 1 positive and 2 negative, in that case you have extension along 1 direction and compression along the 2 perpendicular directions.

So, this is called a uni-axial extension. So, if I i just try to plot it here, there is the sum of 3 axes, I have 1 direction extension and I have fluid coming in in the other 2 dimensions and stretching along the third direction. So, that is called uni-axial extension 1 positive and 2 negatives and on the other hand, if I have 2 positive and 1 negative, it is the other way around, there is stretching along 2 directions and there is extension along the third direction.

So, the would look something like this, we have stretching along 2 directions, there is bi-axial extension, you cannot have all 3 positive, because then the sum of the Eigen vectors would not be equal to 0. So, these are the different kinds of deformation that are represented by this rate of symmetric traceless part uni-axial extension, bi-axial extension or planar extension. And that comes about the fact that, it is a symmetric tensor Eigen vectors are orthogonal and Eigen values the sum of those is equal to 0 to be just traceless. So, these in brief are the types of deformation that can take place and our next step is to relate the stresses to the different deformations that can place, we will briefly review this before, we go on to derive the momentum conservation equations, which depend up on these rates of deformation. So, I will continue in the next lecture.